

Risk Management Study for Managed Futures

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Abstract: This paper describes some statistical aspects to implement the Value at Risk (VaR) for a Fund which is actively trading the Euro Bund Future(FGBL). First, we describe FGBL instrument and review the FGBL statistical properties. As it is usually the case for financial series, the distribution of FGBL returns exhibits fat tails. Different VaR models are estimated: normal VaR, historical VaR, Cornish-Fisher approximation, Extreme Value Theory, Pareto fitting, volatility models (RiskMetrics, GARCH). The Conditional Value at Risk (CVaR) is also studied. VaR is "blind" on actual losses beyond the VaR. The CVaR defined as the expected losses in case the VaR is exceeded. CVaR is also a more consistent risk measure while VaR is not. In presence of fat tails, the CVaR will lead to less risky and more consistent positions.

The different models will lead to different values of risk measures (Var, CVaR, Maximum Drawdowns). The differences between estimated values can be interpreted as a model risk that is minimized by taking the worst case over all models.

The VaR models are backtested using more than 14 years of daily data ranging from Nov 23, 1990 to October 11, 2004.

1 Value at Risk Models

In today's financial world, Value-at-Risk (VaR) has become the benchmark risk measure. VaR summarizes the expected loss over a target horizon within a given confidence interval. In other words, it is the decline in portfolio market value within a given time interval (such as one month = 20 trading days) with a probability not exceeding a given number (such as 1 percent or 5 percent):

$$Prob((\Delta W \leq -VaR_\alpha) = \alpha$$

The Risk Management Policy has set up a the month VaR to 4% of its capital with a 95% confidence.

The leverage is continuously updated to meet this VaR objective. Accurate estimation of the VaR is important. If the Risk Manager underestimates the VaR, then leverage will be smaller than it should be and the fund may get penalized to meet the returns objective. If, however, the Risk Manager over estimates the VaR, the fund will not meet its risk management objective and will presumably gets penalized by investors.

To calculate the VaR, it is necessary to determine the probability distribution of the portfolio value change. We don't need the full probability distribution, only small quantiles are required.

By the very nature of the problem, VaR estimation is highly dependent on good predictions of uncommon events, or catastrophic risk. As a result, any statistical method used for VaR estimation has the prediction of tail events as its primary goal.

Estimation of a small quantile is not an easy task, as one wants to make inference about the extremal behavior of a portfolio, i.e. in an area of the sample where there is only a very small amount of data. Furthermore (and this is important to note), extrapolation even beyond the range of the data might be wanted, i.e. statements about an area where there are no observation at all.

Under the acronym "let the tails speak", statistical methods have been developed only on that part of the sample which carries the information about the extremal behavior, i.e. only the smallest or largest sample values. Such methods are based on Extreme Value Theory, they are not solely based on the data but includes a probabilistic argument concerning the behavior of the extreme sample values. Extreme value theory provides a natural approach to VaR estimation. The key to this approach is the extreme value theorem, which tells us that the distribution of extreme returns converges asymptotically to a particular known distribution. This distribution has three parameters: a mean and standard deviation, and a third parameter, the tail index, which gives an indication of the heaviness of the tails.

Another complication is the fact that most financial time series are not independent, but exhibit temporal dependence structure, which is often captured by volatility models, such as

GARCH models. GARCH models account for volatility fluctuations and clustering: periods of high volatility tend to alternate with periods of low volatility.

GARCH performs better in signaling the continuation of a high risk regime since it adapts to the new situation. The GARCH methodology thus necessarily implies more volatile risk forecasts than the static unconditional approach. Because the GARCH methodology quickly adapts to recent market developments, it meets the VaR constraint more frequently than the static unconditional approach.

Typically, risk volatility from a GARCH model can be 4 times higher than for a static unconditional model (see for example, Danielsson & Vries [4]).

However, GARCH type models (including RiskMetrics) typically perform worse when disaster strikes since the static approach structures the portfolio against disasters, whereas GARCH does this only once it recognizes one has hit a high volatility regime. This is the reason why the unconditional and static VaR models should apply at any time.

An other issue is related to the VaR itself. Artzner, Delbaen, Eber & Heath [1] have criticized VaR as a measure of risk on two grounds. VaR is not a coherent risk measure (non sub-additivity). There are cases where a portfolio can be split into sub-portfolios such that the sum of the VaR corresponding to the sub-portfolios is smaller than the VaR of the total portfolio. This may cause problems if the risk-management system of a financial institution is based on VaR-limits for individual books. Moreover, VaR tells us nothing about the potential size of the loss that exceeds it.

Artzner et al. propose the use of Conditional Value-At-Risk (CVaR) instead of VaR. CVaR is the expected loss given that the loss exceeds VaR. The CVaR measure is also known as the Expected Shortfall, the Tail Conditional Expectation (TCE) or the Expected Tail Loss. In mathematical term, CVaR is given as:

$$CVaR_\alpha = E(\Delta W | \Delta W \leq VaR_\alpha)$$

An other distinct advantage gained from using the CVaR is its relative efficiency of implementation. CVaR can be optimized using linear programming (cf. Rockafellar and Uryasev [16]) which allow handling portfolios with very large numbers of instruments and scenarios. Numerical experiments indicate that the minimization of CVaR also leads to near optimal solutions in VaR terms because CVaR is always greater than or equal to VaR. Moreover, when the return-loss distribution is normal, these two measures are equivalent, i.e., they provide the same optimal portfolio.

As for VaR, CVaR is estimated under different models (Normal, Historical, Extreme Value based estimation, ...).

2 The Euro Bund Future, statistical aspects

Most financial series exhibit fat tails, i.e. the actual VaR is more severe than the normal VaR. The literature on fat tails in finance is quite large. However, it focus mainly on foreign exchange and stocks markets. Indeed, very few papers deal with bonds or bond futures. Perhaps because bonds returns are less volatile than forex or stocks returns, and therefore it is believed that bonds pose less risk than other assets. This is argument is wrong. What matters is not the volatility per se, but the position in that asset relative to capital. Even low volatility asset may become risky if leverage is high (LCTM, for example).

The Fund is actively trading the Euro BUND future. The BUND Future has become the main instrument for hedging long term interest rates in the euro area and therefore is a very liquid instrument.

As described by [Eurex](#), the Euro Bund Future (FGBL) is a notional long-term debt instrument issued by the German Federal Government with a term of 8.5 to 10.5 years and an interest rate of 6 percent. Contract Size is EUR 100 000. Quotation is in a percentage of the par value, carried out two decimal places. Minimum price movement (tick) is 0.01 percent, representing a value of EUR 10.

The data used in this study are perpetual contracts, i.e. the aggregation of the most liquid contracts at the nearest maturity, starting from Nov 23, 1990 up to October 11, 2004. This series represents more than 3500 daily quotes including high, low, open, close and intraday hourly data.

Simple exploratory data analysis should stand at the beginning of every risk analysis.

We perform the study for both returns and simple increments. Since Future contracts are traded in basis points, it is more natural to model the data in terms of increments ($Price_{i+1} - Price_i$), rather than usual log returns ($returns = \log Price_{i+1} - \log Price_i$). There are no statistical evidence to choose one model rather than the other, except that increments are more precise and directly are related to the P&L (1 basis points is EUR 10).

The figures (fig. 2 to (fig. 4) compare the Euro Bund Future returns distribution and increments with its gaussian fit as histogram (fig. 2 & 3) or quantile-quantile plot (fig. 4 & 5). As usual, the tail is not so fat for positive returns. However, the tails are much less fat than stock indexes tails, for example. Both extremes (left and right) looks truncated, there are neither negative returns below -2% (or 200 bps) nor positive returns above 2% (or 200 bps).

2.1 Symmetrization of data

Since dynamic strategy can lead to either short or long positions on the underlying asset, both positive and negative returns should be studied together. We can study either only

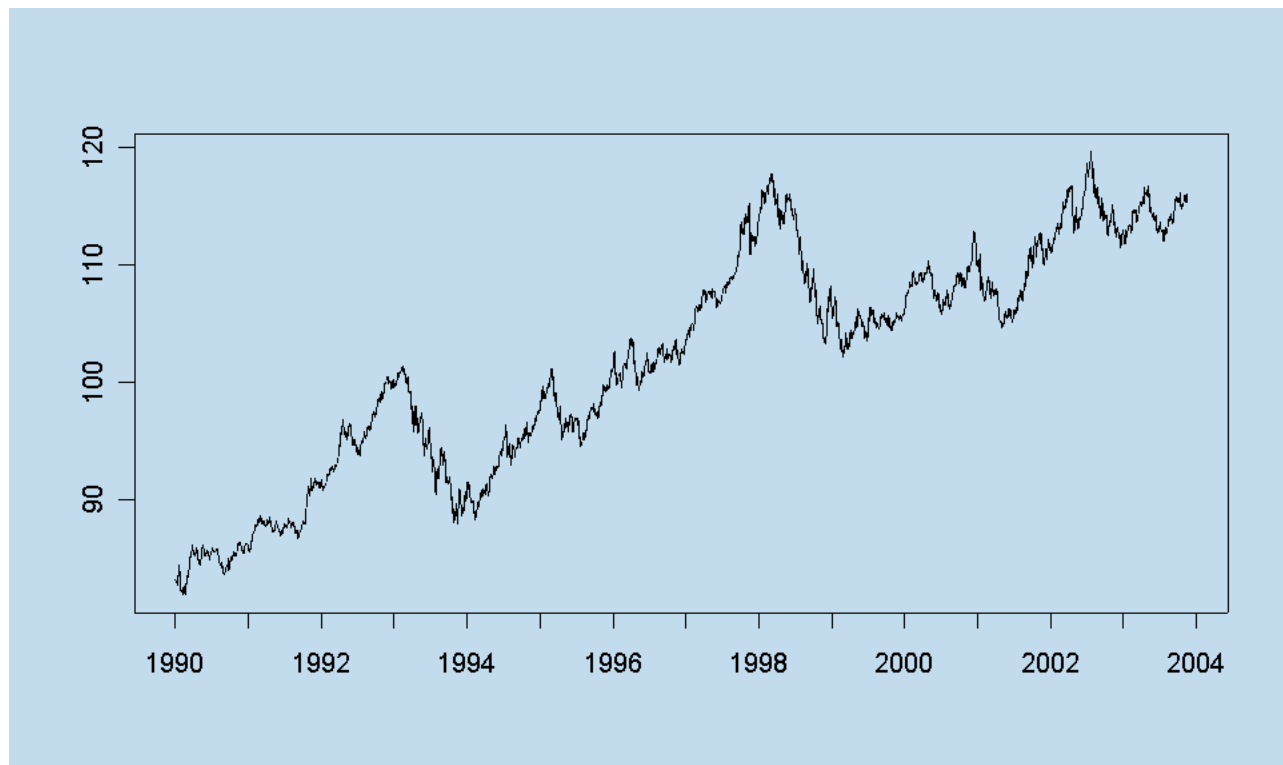


Figure 1: Euro Bund Future prices

positive or negative tails separately (this is also performed). An other way is to build a symmetrical sample as a concatenation of the original returns r_0, r_1, \dots, r_n with its opposite $-r_0, -r_1, \dots, -r_n$. This construction can be used to study the IID case only. However, in day to day operations actual long or short VaR are estimated independently. For temporal dependencies, other methods such as block bootstrapping can be used, as proposed in [3].

The symmetrical distribution (see table 1) has no skewness by construction. The kurtosis is approximatively the same as the original series, $K = 2.37$. The same remarks apply to increments.

The table 1 displays quantiles of the empirical distribution for returns (in percent), increments with their related gaussian fits (a gaussian fit is defined as the normal distribution where the mean and the standard deviation are the same as the sample estimates).

This table clearly shows the fatness of both negative returns and increments. Positive returns and increments look less risky than the gaussian fit. This "stylized" fact can also be shown in a negative skewness. It is clear that the returns of the FGBL contracts are quite different from stocks markets. Bonds Futures are much less volatile and seem to have

smaller tails. The returns seem to be truncated at both tails: absolute daily returns are smaller than 2%, event in the worst market conditions.

	quantile									
	0%	0.1%	1%	5%	10%	90%	95%	99%	99.9%	100%
returns	-1.96	-1.60	-0.984	-0.557	-0.390	0.395	0.525	0.818	1.22	1.91
Gaussian fit	$-\infty$	-1.04	-0.784	-0.552	-0.428	0.447	0.571	0.803	1.06	∞
sym. returns	-1.96	-1.44	-0.890	-0.542	-0.392	0.392	0.542	0.890	1.44	1.96
gaussian fit	$-\infty$	-1.05	-0.794	-0.561	-0.437	0.437	0.561	0.794	1.05	∞
increments	-196	-165	-104	-58	-40	41	54	84	117	174
gaussian fit	$-\infty$	-107	-80	-57	-44	46	58	82	109	∞
sym. increments	-196	-153	-92	-56	-40	40	56	92	153	196
gaussian fit	$-\infty$	-108	-81	-58	-45	45	58	81	108	∞

Table 1: Quantile of FGBL returns, increments and symmetrical empirical distribution - returns are represented as percentage. For example, the probability to lose more than 0.984% is 1%. In other words, the daily VaR at 1% critical level for one contract is 0.984%. Increments are defined in basis points. For example, according to this historical simulation, there is a 1% probability to lose more 104 basis points. returns and increments are quite comparable since the price is about 100. However, increments are more suited since they are directly related to P&L, ie 1bps is 10 euros per contract.

2.2 Sample statistics

Simple sample statistics are computed for both returns or increments, for both original or symmetrical series:

$$\begin{aligned}
 \text{mean} &= E(X) \\
 \text{variance} &= E(X^2) - E(X)^2 \\
 \text{standardDeviation} &= \sqrt{\text{variance}} \\
 \text{skewness} &= \frac{E(X - E(X))^3}{\text{standardDeviation}^3} \\
 \text{excessKurtosis} &= \frac{E(X - E(X))^4}{\text{standardDeviation}^4} - 3
 \end{aligned}$$

Sample statistics exhibit an excess kurtosis and a strong negative skewness, i.e negative returns are more severe (but less frequent) than positive returns:

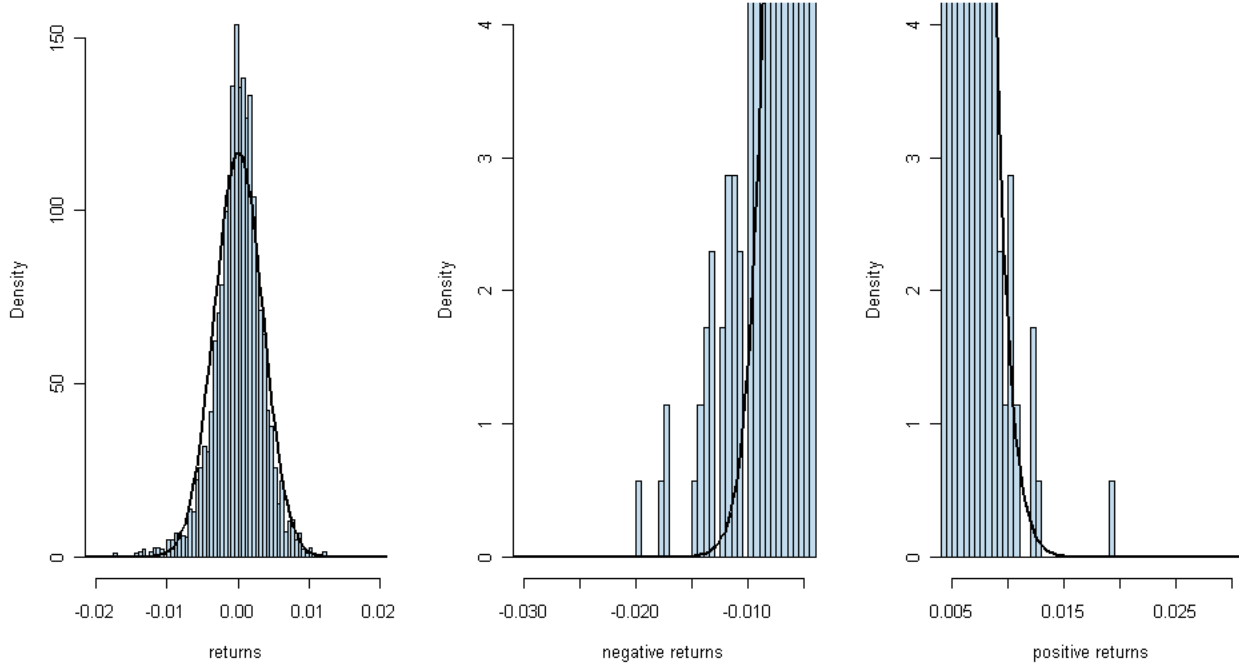


Figure 2: Histogram of Euro Bund Future returns superposed with the gaussian fit.

	mean	standard deviation	skewness	kurtosis
returns	9.53e-05	0.00341	-0.473	2.37
sym. returns	0	0.00341	0	2.32
increments (bps)	0.94	35	-0.516	2.36
sym. increments (bps)	0	35	0	2.30

Table 2: Sample statistics of FGBL for returns and increments (in basis points)

At daily or montly time period, the sample mean is very small compared to the standard deviation or tail estimates. It can be considered as zero.

2.3 Extreme Value Theory

Extreme value theory (EVT) is a powerful method for modeling and measuring extreme risks [11].

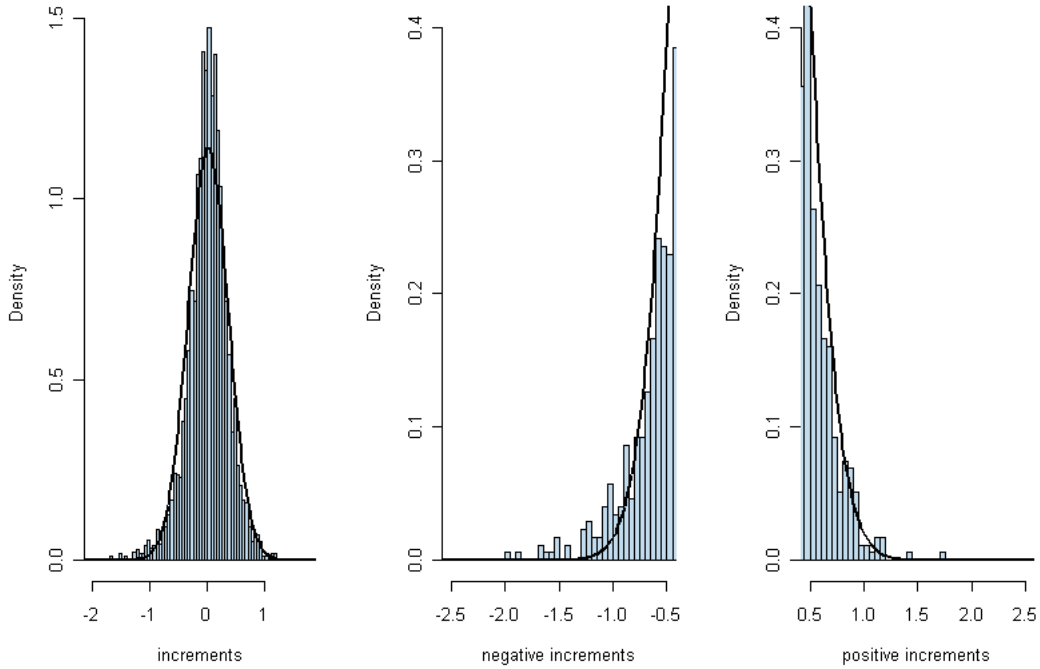


Figure 3: Histogram of Euro Bund Future increments superposed with the gaussian fit.

EVT focuses on the k largest (or smallest) returns or increments. The main result is the Extreme Value Theorem, that is as most as important (even more) than the Central Limit Theorem.

Let X_1, X_2, \dots, X_n be iid random variables (X_i are returns, for example) and $M_n = \max(X_1, X_2, \dots, X_n)$. Then, under general conditions that apply to financial series, there exist constants λ_n, σ_n and a limiting distribution H such that

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n - \lambda_n}{\sigma_n} \leq x\right) = H(x)$$

where $H(x)$ has the form

$$H(x) = \exp(-(1 + \xi x)^{1/\xi})$$

H is the Frechet distribution if $\xi > 0$, the Gumbel distribution if $\xi = 0$ or the Weibull distribution if $\xi < 0$.

Most of financial assets belong to the Frechet domain ($\xi > 0$) corresponding to fat tail case with tail index $1/\xi$ (Gumbel and Weibull forms are not fat tailed).

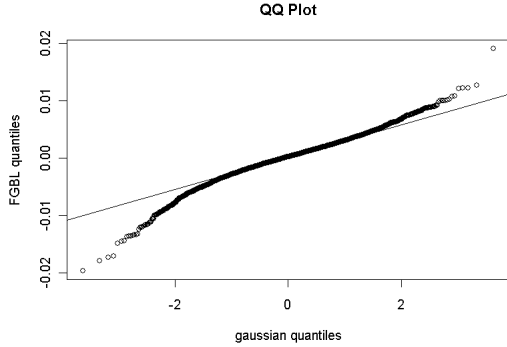


Figure 4: QQ plot of Euro Bund Future returns

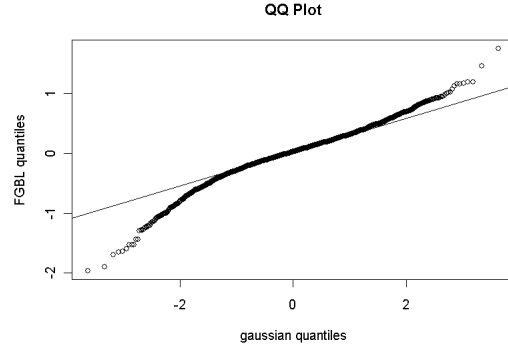


Figure 5: QQ plot of Euro Bund Future increments

An intuitive interpretation of fat tails is the following: the sum S_n of the random variables tends to be determined by its maximum values:

$$\lim_{x \rightarrow \infty} \frac{P(S_n > x)}{P(M_n > x)} = 1$$

In other the actual loss over a period is determined by very few days ...

The most suited tool to study Extreme Values are the Peaks-Over-Threshold (POT) models. POT models observations which exceed a high threshold. The POT models are generally considered to be the most useful for practical applications, due to their more efficient use of the (often limited) data on extreme values. They provide a simple tool for estimating measures of tail risk and deliver useful estimates of Value-at-Risk (VaR) and CVaR.

There are the semi-parametric models built around the Hill estimator and the fully parametric models based on the generalized Pareto distribution or GPD.

Hill estimator is a maximum likelihood of power tails:

$$P(X < -x) \approx Cx^{-\delta}$$

δ is the tail index. Moment of order k are only defined for $k < \delta$, moments of order $\geq \delta$ are infinite. If $\delta < 4$, the kurtosis is infinite. A normal distribution has an infinite tail index, all moments are defined at any order.

The figure 6 shows a Hill estimation of daily increments for FGBL. The tail index seems to be in the range of 3-6. This result is consistent with the other studies on FGBL. In [18], T Werner and C. Upper studied the High Frequency returns of BUND

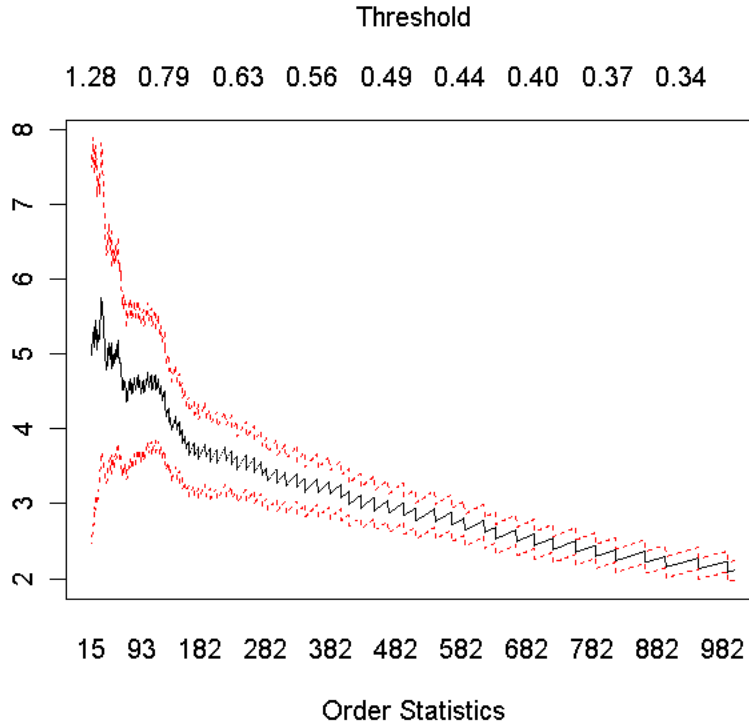


Figure 6: Hill estimation the Tail Index of FGBL increments with 0.95 confidence bands.

Future. The data set includes five minutes prices of the Bund future from January 1997 to December 2001. Using the Hill estimator, they found fat tails at all sampling intervals with a tail index ranging from 3 at five minutes sampling up to 5-6 for the right tail (positive returns) of daily returns.

sampling frequency	5 minutes	1 hours	1 day
left-right tail index	3.01-3.33	3.36-3.90	4.55-6.28

Table 3: Left-Right Tail index of the BUND Future, from Werner & Upper [18]. The left tail (negative return) index is smaller than the right tail (positive returns), a well known stylized fact related to strong negative skewness, with more severe negative returns than positive returns. The confidence interval is increasing with the time horizon and become quite large for daily estimates. For example the 95% confidence interval for the daily right tail is 4.61-10.18. The left tails, corresponding to negative returns, tend to be slightly thicker than the right tails irrespective of the frequency.

With the aid of a recently developed test for changes in tail behavior, they have identified several breaks in the degree of heaviness of the log-return tails. Such breaks were particularly pronounced during 1998 and 2001, probably in relation with the Russia and LTCM crises in the former, and the September 11 attacks in the latter year.

The behavior of the tails of a distribution is not necessarily captured by measures for volatility. For example, in 2000 volatility declined, suggesting a reduction in risks, whereas the probability of extreme price changes, as measured by the tail index, actually increased. This shows that the tail index contains important information for financial market risk assessment beyond that captured in standard volatility measures.

The Generalized Pareto Distribution fitting gives contradictory results.

In a GPD fit, we are interested in modeling the distribution over a predefined threshold u .

$$F_u = P(X - u \leq y | X > u)$$

For the distributions verifying the Extreme Value Theorem (ie most of commonly used distributions) and for a large enough threshold, there exist 2 constants: ξ , the inverse of the tail index and β , a scaling parameter, such that F_u converge to the Generalized Pareto Distribution:

$$G_{\xi\beta} = \begin{cases} 1 - \left(1 + \xi \frac{x}{\beta}\right)^{-1/\xi} & \text{for } \xi \neq 0 \\ 1 - e^{-\frac{x}{\beta}} & \text{for } \xi = 0 \end{cases}$$

if $\xi > 0$, then $G_{\xi\beta}$ is the Pareto distribution

One distinctive advantage of the GPD is that one can have VaR and CVaR in closed forms. If n is total sample size and n_u the number of returns exceeding the threshold u then

$$F = 1 - \frac{n_u}{n} \left(1 + \xi \frac{x - u}{\beta}\right)^{-1/\xi}$$

and

$$VaR_\alpha = u + \frac{\beta}{\xi} \left(\left(\frac{n}{n_u} \alpha \right)^{-\xi} - 1 \right)$$

One can also estimate the the Expected Shortfall or CVaR, i.e., the expected loss in case of exceeding the VaR:

$$ES_\alpha = VaR_\alpha + E[X - VaR_\alpha | X > VaR_\alpha]$$

$$\frac{ES_\alpha}{VaR_\alpha} = \frac{1}{1 - \xi} + \frac{\beta - \xi u}{(1 - \xi) VaR_\alpha}$$

Applying, this result to the FGBL data, the GPD fitting seems to give an infinite tail index, hence no fat tails. This result is not consistent with the fat tail behavior and may be due to the truncations effects of the FGBL distribution. For one contract, the GPD fit provides the following risk measures:

p	VaR	CVaR
0.9500	0.56	0.79
0.9900	0.93	1.17
0.9990	1.50	1.76
0.9999	2.11	2.40

The results are very similar to the historical estimates (both for VaR 3.2 and CVaR 3.4.2), with the difference that GPD needs much less data to provide consistent and precise estimates. The main benefits of using Extreme Value Theory is in its capability to get consistent and robust results for very small quantiles where very few data are available. However, in this typical case, we are mainly considering the 95% quantile, an area where the EVT will not provide a distinctive advantage. In addition, the FGBL returns are not so fat tailed as other financials returns such as stock markets.

3 The Static VaR

3.1 The Normal Approximation

The normal (gaussian) VaR estimates has the main advantage of being very simple. It can be used as a first approach to provide quick (but dirty) rough estimates. However, as we will see later, the main problem of the normal model is that it is wrong and can dangerously underestimate the risk.

In a gaussian world, the VaR scales as the square root of the time horizon. For example:

$$VaR(10days) = \sqrt{10}VaR(1day)$$

For a multivariate model; i.e. a portfolio with weights w_i for asset i and correlation ρ_{ij} between assets i and j $i, j = 1 \dots q$,

$$VaR(portfolio) = \sqrt{\sum_{i,j=1}^q \rho_{ij} w_i w_j VaR_i VaR_j}$$

Consider the simple example of a portfolio invested in a single risky asset, the Euro Bund Futures (FGBL). Let w be the weight for FGBL. With a single risky asset, w is also the

leverage. The remaining proportion $w_{riskfree} = 1 - w$ is invested in the short term risk free rate (typically, EONIA) or borrowed if $w > 1$. In this simple case, the VaR reduces to

$$VaR(portfolio) = wVaR(FGBL)$$

Under a gaussian hypothesis, this VaR is simply:

$$VaR(portfolio; T, 1 - p) = \Phi^{-1}(p)wW\sqrt{T}\sigma$$

where $\Phi^{-1}(p)$ is the normal quantile and σ the daily volatility of returns. The volatility for T days is extrapolated from the one day volatility, implementing a "square root of time" rule which implies that returns are normal with no serial correlation. However, for fat tailed data, a $T^{1/\beta}$ is more appropriate, where β is the tail index.

For example,

$$VaR(portfolio; T, 95\%) = -1.65wW\sqrt{T}\sigma$$

$$VaR(portfolio; T, 99\%) = -2.33wW\sqrt{T}\sigma$$

Consider, for example, a risk management objective defining the one month (20 trading days) 95% VaR to be 4% of the funds capital, the weight in risky asset should be adjusted such that

$$w = \frac{0.04}{1.65\sigma\sqrt{T}}$$

An other approach is to consider increments and number of contracts rather than returns and weight.

Consider a long position in N contracts. $\Delta W = N \times M \times \Delta Price$, where M is the contract multiplier.

In case of the FGBL contract, the multiplier is 1000, i.e, 1 basis point (0.01 in price) represents 10 euros. One contract value is equivalent to an investment of $M * Price$ euros.

If we model the increments $\Delta Price$ as a gaussian (that is not less false than than considering the returns $\Delta \log Price$), then ΔW can be approximated by a normal distribution with the standard deviation is $NM\sigma$, where σ is the standard deviation of increments. Then the VaR over a time horizon T becomes:

$$VaR(portfolio; T, 5\%) = -1.65NM\sigma\sqrt{T}$$

$$VaR(portfolio; T, 1\%) = -2.33NM\sigma\sqrt{T}$$

Conversely, the number of contracts can be determined from the VaR objective:

$$N_{0.05} = \frac{-VaR(portfolio; T, 5\%)}{1.65M\sigma\sqrt{T}}$$

The weight (or leverage) can be recovered from its definition, i.e:

$$w = \frac{N * M * Price}{W} = \frac{AssetValue}{PortfolioValue}$$

where W is the portfolio value. Or having defined w , we can determine the maximum number of contracts N that the trader is authorized to buy or sell by:

$$N = \left\lfloor \frac{wW}{MP} \right\rfloor$$

where $\lfloor x \rfloor$ is the integer part of x (rounding to the nearest smaller integer)

Consider, for example, a portfolio of 10 Millions euros and a risk management policy so that the leverage is continuously adjusted to meet a maximum portfolio VaR to be less than 4% over a month at 95% confidence level. The typical daily volatility of the FGBl increments is about $\sigma = 35$ in basis points.

Under the normal hypothesis, the number of contracts shall verify

$$N_{0.05}^{gaussian} = \frac{0.04 * 10,000,000}{1.65 * 1000 * 0.35 * \sqrt{20}} = 155$$

The size of the position (short or long) shall not exceed 155 contracts.

Assuming a price of 117, then the portfolio value in risky asset is $N * M * Price = 18,252,000$, the leverage is $w = 18,252,000/10,000,000 = 1.82$.

If we consider a 99% VaR, then the number of contracts should not exceed

$$N_{0.01}^{gaussian} = \frac{0.04 * 10,000,000}{2.33 * 1000 * 0.35 * \sqrt{20}} = 109$$

3.2 Historical Simulation

Since dynamic strategy are either short or long on the underlying asset, both positive and negative returns should be studied together. From the symmetrical FGBl increments quantile (see table 1), the daily VaR at 95% for one contrat (long or short) is about 56 basis points.

Hence, according to historical quantile, the objective of 1 month VaR (at 95% confidence level) not exceeding 4% of wealth is fulfilled with a number of contracts N :

$$N_{0.05}^{hist} = \frac{0.04 * 10,000,000}{1000 * 0.56 * \sqrt{20}} = 160$$

The size of the position (short or long) shall not exceed 160 contracts.

Assuming a price of 117, then the portfolio value in risky asset is $N * M * Price = 18,687,140$, that is a leverage $w = 1.86$.

This is slightly higher than the gaussian hypothesis.

If we consider a 99% VaR, the 1% quantile is -92 basis points, and the number of contracts should not exceed

$$N_{0.01}^{hist} = \frac{0.04 * 10,000,000}{1000 * 0.92 * \sqrt{20}} = 97.2$$

This is less risky than the gaussian hypothesis (110 contracts).

3.3 Cornish Fisher Approximation

Under the normal hypothesis both skewness and excess kurtosis should be equals to zero. In fact, as shown in table 2, the increments (or returns) distribution exhibit a strong negative skewness and a kurtosis in excess, that is a very common stylized fact. One can use a Taylor expansion of the normal quantile to take into account the skewness and kurtosis:

$$z \approx z_0 + \frac{1}{6}(z_0^2 - 1)S + \frac{1}{24}(z_0^3 - 3z_0)K - \frac{1}{36}(2z_0^3 - 5z_0)S^2 \quad (1)$$

where z_0 is the $1 - \alpha$ quantile of the normal distribution: $N(z_0) = 1 - \alpha$, S is the skewness and K the kurtosis in excess. note that if the skewness is null, the approximation reduced to the simple correction:

$$z \approx z_0 + \frac{1}{24}(z_0^3 - 3z_0)K$$

Then Cornish Fisher VaR is

$$VaR(\alpha, T) \approx -z\sigma\sqrt{T}$$

for each euro of risky asset.

The table 4 reports the Cornish Fisher correction z for the FGBL, with *skewness* = -0.51 and *kurtosis* = 2.36 , as well as the corresponding number of contracts.

	0.01	0.05
Cornish Fisher (Gaussian)	2.4 (2.33)	1.44 (1.65)
N contracts	106	177

Table 4: Cornish Fisher Approximation

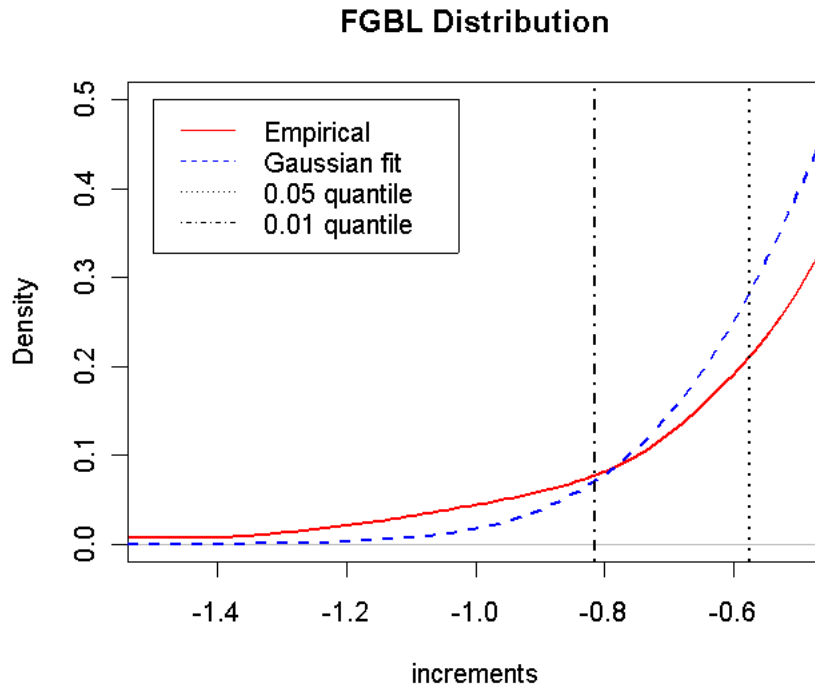


Figure 7: Empirical vs Normal density

This figure displays the empirical distribution (estimated via a nonparametric based estimator) versus its Normal fit. For the 0.05 quantile, the empirical distribution curve is under the gaussian fit, i.e. the empirical VaR is smaller than the normal VaR at 0.05 critical level. For the 0.01 quantile, the empirical distribution is slightly above the gaussian fit, i.e. the empirical VaR is greater than the normal VaR at 0.01 critical level.

The conclusion is similar to the Historical VaR, i.e. the Cornish Fisher VaR at 0.05 (resp 0.01) level leads to more (resp. less) contracts than the normal hypothesis. It means that

the return distribution is crossing the gaussian fit between 0.05 and 0.01 (see figure 7).

For very small quantile, the Cornish Fischer approximation is questionable since it is based on of the kurtosis (fourth moment) that may not exist if tail index is smaller than 4, or if it exists, its sample estimates may be highly imprecise, the variance of the kurtosis estimator depends on the moment of order 8 that is likely to be infinite.

3.4 Conditional VaR

3.4.1 Introduction

VaR is "blind" toward risks that create large losses beyond the VaR. If the VaR objective is used as a trading limit, then traders may have an incentive to run strategies that exactly generate such type of risks:

- Increase in bets until a certain profit is reached (the classical doubling strategy).
- Buy defaultable bonds and sell risk less bonds (LCTM).
- Sell far out of the money put options.
- Sell insurances for rare events.

1

Optimizing at VaR (rather than CVaR) boundaries may be misleading and very dangerous. Optimization under VaR/CVaR constraints will be addressed in other papers.

VaR and CVaR can be compared under a normal model. For example, for the normal distribution $N(0, 1)$, the CVaR relates to the VaR as follow:

$$CVaR_\alpha = -\frac{\varphi(z)}{\alpha}$$

where φ is the normal density:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

¹Note: VaR can also be interpreted as the cost of a digital put option on the value of the fund. The Expected Shortfall could be interpreted as the price of European put option with a strike at the VaR level. Options valuation technique can be applied to risk measurements and vice versa. Where risk measurements techniques are well suited to measure tail behavior, they are also specially suited to value far out the money options.

and

$$VaR_\alpha = -z$$

At 0.05 critical level, the ratio between CVaR and VaR is 1.254. For a 0.01 level, this ratio is only 1.146.

$$\begin{aligned} CVaR_{0.05} &= 1.254 VaR_{0.05} \\ CVaR_{0.01} &= 1.146 VaR_{0.01} \end{aligned} \tag{2}$$

This result can be used to translate VaR into CVaR objectives and vice versa. For example, the previous one month 4% $VaR_{0.05}$ objective can be turned into a normal 5% = 4 * 1.25 $CVaR_{0.05}$ objective. At 0.01 level, the 4% $VaR_{0.01}$ can be turned into a 4.58% $CVaR_{0.01}$ objective.

Note that when α becomes small, z becomes large, and the Mill's approximation applies, i.e

$$P(X < -z) \approx \frac{\varphi(z)}{z}$$

So that, when α becomes small CVaR and VaR becomes similar and the previous conversion ratio tends to 1 as α tends to zero:

$$CVaR_\alpha = -\frac{\varphi(z)}{\alpha} \approx -z = VaR_\alpha$$

For power tails distributions $F(x) \sim x^{-\beta}$, the situation is quite different. There is a constant ratio between the VaR and the CVaR:

$$\begin{aligned} VaR_\alpha &= \alpha^{-1/\beta} \\ CVaR_\alpha &= \frac{\beta}{\beta - 1} VaR_\alpha \end{aligned}$$

At the extreme, any level of CVaR can be reached by the "Peso problem strategies" (see Taleb [17]). Consider the trade with the following payout:

$$X = \begin{cases} 1 + 2p(x - 1) & \text{with probability } 0.5 \\ -1 & 0.5 - p \\ -x & p \end{cases}$$

with $p < \alpha < 0.5$ and $x > 1$. So that, with any probability $\alpha > p$, we cannot loss more than -1, hence, $VaR_\alpha = -1$ Letting x tend to ∞ , the VaR remains constant = -1, while the CVaR is unbounded.

$$X|X \leq -1 = \begin{cases} -1 & \text{with probability } 1 - 2p \\ -x & 2p \end{cases}$$

and $CVaR = E(X|X \leq -1) = (1 - 2p) * (-1) + 2p * (-x) = -1 - 2p(x - 1)$

This simple example shows that we should be very cautious in defining risk measures. To optimize strategies at the VaR boundaries, the VaR objective is converted into an equivalent CVaR to avoid the "Peso problem". Then optimization will be performed under the CVaR constraints rather than VaR.

Recall that the VaR is blind on losses that may actually happen if the VaR is met. hence, trading at VaR boundaries may be very risky in presence of fat tails.

However, for even more general case, whatever the distribution, there are universal bounds of VaR and CVaR for a given value of mean and standard deviation This allow us to compute VaR and CVaR even when the distribution is unknown! The bounds are listed in the table 5 for some critical levels. The maximum Expected Shortfall (or CVaR) at level α and volatility σ is

$$CVaR_\alpha = \sqrt{\frac{1-\alpha}{\alpha}}\sigma$$

The VaR bound is the same as the CVaR bound.

α	0.1	0.05	0.01	0.005
normal $\varphi(z)/\alpha$	1.76	2.06	2.67	2.89
worst case $\sqrt{\frac{1-\alpha}{\alpha}}$	3	4.36	9.95	14.1

Table 5: Comparison of CVaR under normal distribution and under worst case distribution

This bound may justify the apparently arbitrary Basle rule that consists in multiplying the normal VaR by 3 or 4 (depending on historical VaR) . However, as notices by Danielsson and De Vries those bounds are not tight and actual VaR is not so close to the worst case.

A natural estimator of CVaR is the following. Sorting the returns (or increments) in increasing order $r_1 \leq r_2 \leq \dots \leq r_N$, then an estimator of CVaR is:

$$CVaR_\alpha \approx \frac{1}{K} \sum_{i=1, K} r_i \quad (3)$$

with $K = \lceil \alpha N \rceil$.

3.4.2 Application to Fund Management

The CVaR is estimated under different models: the normal model, historical CVaR and the Extreme Value CVar.

	normal VaR and equivalent CVaR	Historical VaR	Historical CVaR
0.05 level	155	160	141
0.01 level	109	97	87

Table 6: Number of contracts under static VaR or CVaR constraint. For example, at the 5% critical level, the objective of 4% VaR over one month is achieved with 160 contracts. The related CVaR objective is set at not losing 5% = 4 * 1.25 of the portfolio value in case of the 4% VaR is exceeded. According to historical data, this objective can be achieved with a number of contracts not exceeding 141

First, recall from equation 2, that the one month 4% $VaR_{0.05}$ objective can be turned into a 5% = 4 * 1.25 $CVaR_{0.05}$ objective (under normal hypothesis). For a portfolio value of 10 000 000 euros, This 5% CVaR will lead to the same number of contracts.

The historical $CVaR_{0.05}$ for FGBL symmetrical increments is -0.7924 . The related number of contracts for a 5% CVaR objective is

$$N_{0.05}^{cvar} = \frac{0.05 * 10,000,000}{1000 * 0.7924 * \sqrt{20}} = 141$$

The size of the position (short or long) shall not exceed 141 contracts.

With a $alpha = 0.01$, the CVaR for one contract and one day is about -1.1764 . The number of contracts is given by:

$$N_{0.01}^{cvar} = \frac{1.146 * 0.04 * 10,000,000}{1000 * 1.1764 * \sqrt{20}} = 87$$

At 95% confidence level, the FGBL contract exhibits a light "Peso Strategy" effect. From the historical data, the VaR objective may authorize 160 contracts, while the normal VaR is 155 contracts and the equivalent CVaR is 141 contracts. In other words the historical VaR is "blind" on actual losses that may occur in case the VaR is met. However this "Peso Strategy" effect is very limited, it disappears at higher confidence level (99% for example).

4 Volatility Models

4.1 Naive estimators

The figure 8 represents the FGBL historical volatility using the a rolling estimation of the variance:

$$\sigma_t^2 = \frac{1}{29} \sum_{k=1,30} (r_{t-k+1} - \langle r \rangle_t)^2$$

with

$$\langle r \rangle_t = \frac{1}{30} \sum_{k=1,30} r_{t-k+1}$$

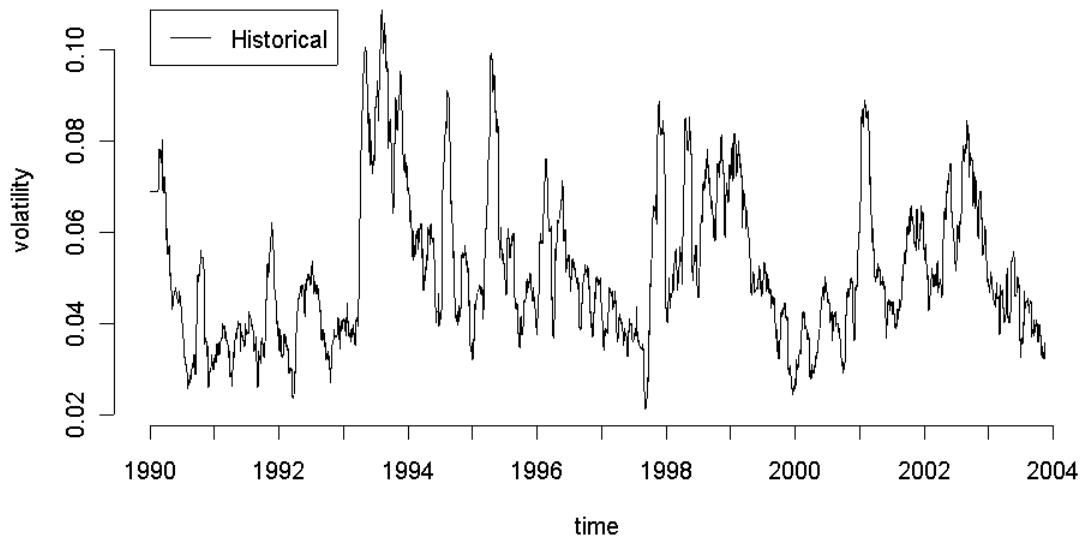


Figure 8: Euro Bund Future Volatility

Value-at-Risk analysis is highly dependent on extreme returns or spikes. The empirical properties of the spikes, are not the same as the properties of the entire return process, including the volatility.

Historical volatility clearly exhibits clustering, periods of high volatility alternate with periods of low volatility. In other word, there exist positive a significant positive serial

correlation in volatility of returns. As a result volatilities can be relatively well predicted with a parametric model such as GARCH or Exponential Moving Average such as the RiskMetrics model.

If, however, one focuses only on spikes, the dependency seems to be reduced. The increments (see figure 9) display no evidence of such clear clustering.

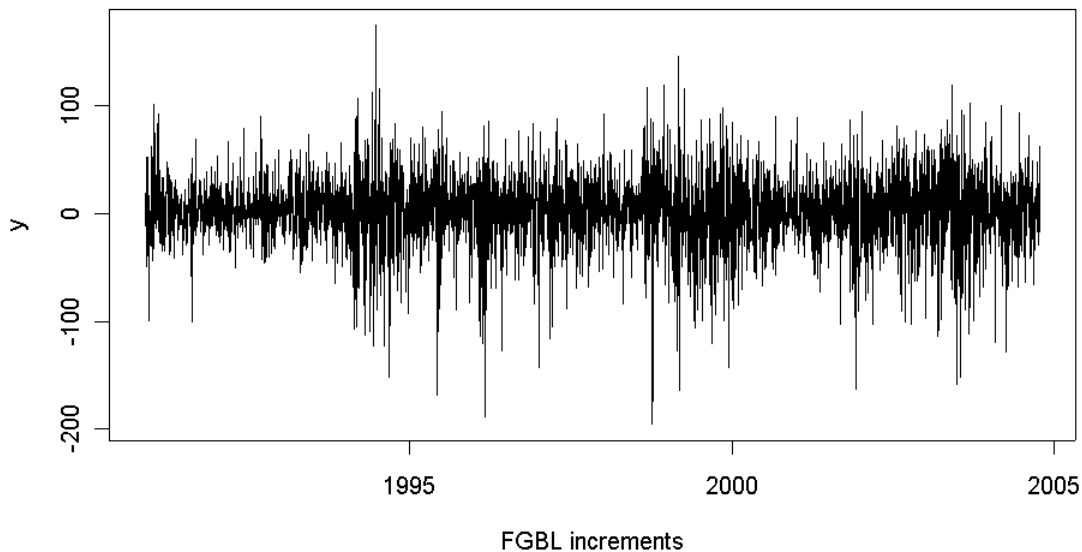


Figure 9: Euro Bund Future Increments

4.2 High/Low based volatility estimators

One of the main challenge in volatility models is to detect change in regime as quickly as possible, and shorten the lags as much as possible. However, if the lags are too short, the estimation becomes noisy. A trade off should be performed. In order to react as quickly as possible to market changes, one can has to use intraday volatility or use highs and lows. It is well known that using highs and lows improves the efficiency by a factor of 5 to 6 compared to sample estimation using close prices only, i.e., 5 to 6 less data are needed to get the same precision. For example rather than using 60 days to estimate the historical volatility, 10 days lag are enough to get the same precision. Hence, for a given precision, High/Low volatility based estimation are much more reactive to regime switch in volatility.

Such estimator are very close to the popular Average True Range. We consider the Parkinson [12] and Roger Satchell [15] estimators:

$$\sigma_{Parkinson}^2 = \frac{1}{N * 4 * \log(2)} \sum_{k=i, i-N-1} (h_k - l_k)^2$$

$$\sigma_{RS}^2 = \frac{1}{N} \sum_{k=i, i-N-1} (h_k - o_k)(h_k - c_k) + (l_k - o_k)(l_k - c_k)$$

where $h = \log(\text{high})$ $l = \log(\text{low})$ $o = \log(\text{open})$ $c = \log(\text{close})$

For the different estimators based on highs and lows, we can consider $N = 10$. This is comparable to a simple estimate with $N = 50$ returns based on closing price.

The figure 10 represents some typical behavior of the different volatility estimators. It can be seen that the High/Low based estimators react more quickly to changing conditions. The RiskMetrics estimators seems too smooth. The highs and lows convey more information about the actual volatility. A closer look at the price during this period (see figure 11)

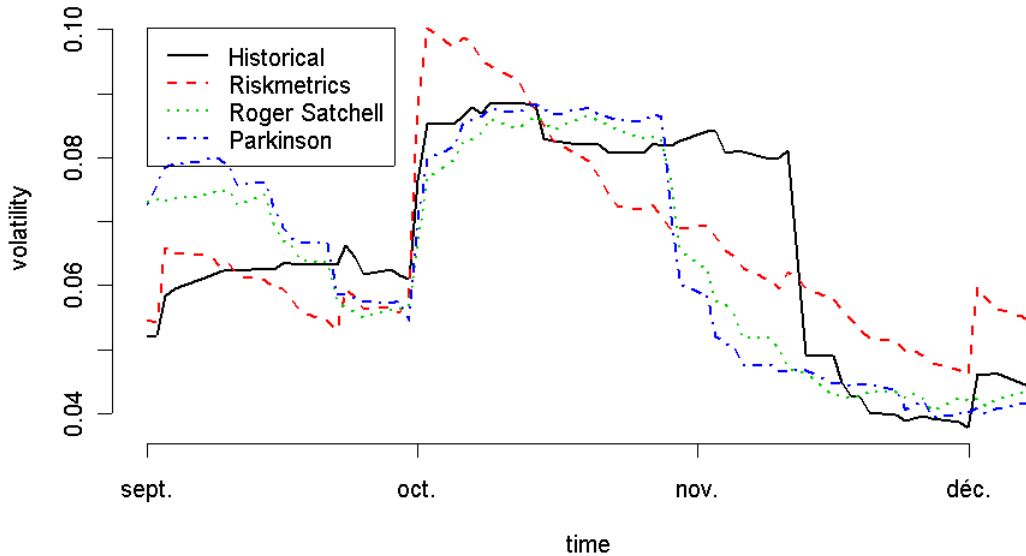


Figure 10: Euro Bund Future Volatilities estimators

shows sharp drop of more than 400 bps in few days. Then the market seems to recover its

normal level of volatility. This behavior is very common and can be better modeled with jumps. However, the High/Low based estimators react more quickly to both the "spike" and the return to a more quiet regime.

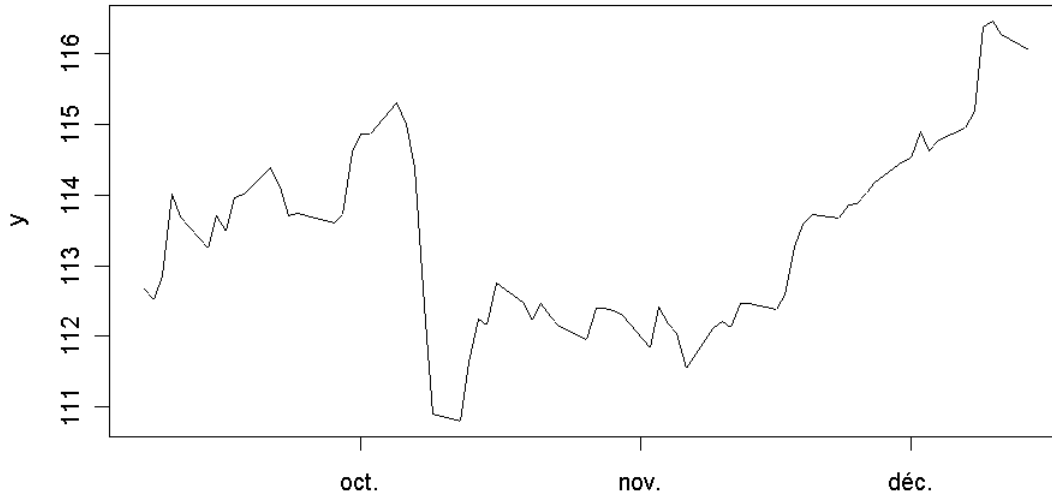


Figure 11: Euro Bund Future Volatilities estimators

4.3 RiskMetrics

RiskMetrics approach was developed in 1994 by JP Morgan [13]. It uses an Exponential Weighted Moving Average (EWMA) of volatility to forecast the time-varying risk. Formally, the variance forecast at time t is a weighted average of the previous forecast, using weight λ , and of the latest squared innovation, using weight $1 - \lambda$:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2 \quad (4)$$

The parameter λ is called the decay factor. The classical values defined by RiskMetrics is $\lambda = 0.94$ for a one day volatility and $\lambda = 0.97$ for a one month volatility. The effective number of days can be derive from the decay factor. 99.9% of the information is contained in the last $\log(0.001)/\log \lambda$ days. For example, if we use $\lambda = 0.94$, then 99.9% of the information

is contained in the last 112 days. For $\lambda = 0.97$, 99.9% of the information is contained in the last 227 days. Hence, the monthly $\lambda = 0.97$ EWMA is smoother than the daily $\lambda = 0.94$. However, the recommended RiskMetrics decay values are defined as an average over many different assets. Decay factors may depend on a specific asset.

If $\sigma_t(\text{riskmetrics})$ is supposed to predict the volatility then we can assess the goodness of prediction by the Root Mean Square Error of prediction:

$$RMSE(\lambda) = \sqrt{\frac{1}{T} \sum_{i=1}^T (r_i^2 - \sigma_i^2(\lambda))^2}$$

Assuming that the daily mean return is closed to zero. The best λ will minimize this RMSE. This is the standard procedure used by RiskMetrics. We performed the same optimization procedure for the FGBL contract and some Stock Indexes for the purpose of comparison (see table 7). The RMSE for the FGBL is 10 times smaller than the RMSE for stock indexes. We verified that decay factors are very close to the RiskMetrics recommended values. However, even after optimization, in sample tests show that it is very difficult to predict the next squared returns. The prediction error ($2.26e - 05$) is about 2 times higher than the variance ($1.16e - 05$) of the parameter that we are supposed to predict. However some predictability still exists. It will greatly enhance the VaR predictions compared to a static and unconditional model.

Note that the FGBL contract has an annualized volatility of 5.42%, that is much lower than the DAX stock index volatility (about 16%), for example.

	FGBL	DAX	CAC	FTSE	SMI
lambda*	0.969	0.947	0.980	0.967	0.934
RMSE	2.26e-05	2.13e-04	2.16e-04	1.01e-04	1.81e-04
variance	1.16e-05	1.06e-04	1.22e-04	6.33e-05	8.56e-05
volatility	3.41e-03	1.03e-02	1.10e-02	7.96e-03	9.25e-03
lag	222	126	335	204	102

Table 7: Optimal RiskMetrics (EWMA) Volatility decay factors. The optimal decay factors lambda* minimize the RMSE. The daily variance and volatility are also computed, as well as the number of lagged days containing 99.9% of the information in the EWMA.

4.4 GARCH Models

GARCH models were introduced by the seminal works of Engle (1982) [6] and Bollerslev (1986) [2]. These models tried to explain several empirical findings of financial market

series. The main innovation was in the modelisation of the conditional variances that were structured with a time-dependent relation. The model can be represented with a set of equations:

$$r_t = \mu(I_{t-1}) + z_t \sigma_t$$

with z_t a residual IID standard variable

$$E[z_t | I_{t-1}] = 0 \quad E[z_t^2 | I_{t-1}] = 1$$

where I_{t-1} represents the information available at time $t - 1$. The standardised residual are coherent with a standardised normal distribution, however other assumptions can be made, including the Student distribution. For the sake of presentation, we will assume $\mu(I_{t-1}) = 0$. Finally, the conditional variance is defined as

$$\sigma_t^2 = a_0 + a_1 r_{t-1}^2 + a_2 r_{t-2}^2 + \dots + a_q r_{t-q}^2 + b_1 \sigma_{t-1}^2 + \dots + b_p \sigma_{t-p}^2$$

The GARCH(1,1) model is most commonly used:

$$\sigma_t^2 = a_0 + a_1 r_{t-1}^2 + b_1 \sigma_{t-1}^2$$

The expected value of variance at time $t + k$ is an autoregressive process:

$$E[\sigma_{t+k}^2] = a_0 + (a_1 + b_1) E[\sigma_{t+k-1}^2]$$

The parameter $\lambda = a_1 + b_1$ can be interpreted as the mean reverting parameter. $1/(1 - \lambda)$ is the mean time to recover a previous level after a shock.

For the FGBL contract, the GARCH(1,1) has the following parameters:

$$a_0 = 1.330e - 07 \quad a_1 = 0.04228 \quad b_1 = 0.9463$$

The unconditional variance is

$$E[\sigma_t^2] = \frac{a_0}{1 - (a_1 + b_1)}$$

i.e, we recover the unconditional volatility $\sqrt{E[\sigma_t^2]} * \text{sqrt}(252) = 5.42\%$.

The mean reverting factor is $\lambda = 0.04228 + 0.9463 = 0.98858$, hence the volatility cycle length is about 87.6 days.

There are also many other GARCH models: asymmetric GARCH, IGARCH, FIGARCH, t-GARCH, ...

4.5 Volatility predictability

In this section, we will assess the different volatility models with respect to their capabilities to actually predict the volatility. To assess the predictability, one can perform a simple linear regression of r^2 versus the predicted volatility:

$$r_t^2 = \alpha + \beta\sigma_t^2 + \epsilon_t$$

The results are presented in the following table:

	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
alpha	-3.801e-06	-5.742e-06	-4.884e-06	-5.477e-06	-1.155e-05
beta	4.819e-03	5.314e-03	5.904e-03	5.948e-03	6.967e-03
alpha std error	1.306e-06	1.369e-06	1.318e-06	1.323e-06	1.774e-06
beta std error	3.875e-04	4.001e-04	4.486e-04	4.380e-04	5.191e-04
alpha t stat	-2.912e+00	-4.195e+00	-3.705e+00	-4.141e+00	-6.509e+00
beta t stat	1.243e+01	1.328e+01	1.316e+01	1.358e+01	1.342e+01
R2	4.251e-02	4.822e-02	4.739e-02	5.031e-02	4.920e-02

The regressions are significant for all models, at least for β . However, the predictability is low as denoted by small R square. The best models are the Parkinson and GARCH(1,1), then RiskMetrics. The naive estimator (sample mean on the last 30 days) is the worst estimator.

5 Volatility based VaR estimation

Value-at-Risk is defined as the maximum amount of loss a portfolio can incur in with a given level of confidence and in a fixed interval of time. Formally it can be represented as a quantile

$$\int_{-\infty}^{VaR_{m,t}(p,k)} f_{m,t+k}(x) = p$$

where p indicates the confidence level of the quantile, m indicates the model, k refers to the horizon of the possible losses, t is the time to which the Value at Risk refers. All these informations are also summarized with the notation $VaR_{m,t}(p,k)$. Usually, the horizon is restricted to $k = 1$, $k = 10$, (the two levels considered by Basel accord) and $k = 20$ for the one VaR. Various alternative models are available for the evaluation of VaR bounds. Within

this paper we focus on a particular class, the volatility based models, including GARCH-type models, RiskMetrics, ... In particular, assuming also that the standardised residuals are normally distributed the VaR can be represented as:

$$VaR_{m,t}(p, k) = \Phi^{-1}(p)\hat{\sigma}_{t+k,m}$$

where $\Phi^{-1}(p)$ is the quantile of the of a standardised normal variable and $\hat{\sigma}_{t+k,m}$ represents the forecast of the conditional variance obtained by model m at time t with an horizon k.

The figure 12 illustrates the inverse relationship between volatility and maximum number of contracts (or leverage). When the volatility is high the number of contracts shall be reduced. Conversely, when the volatility is low, the number of contracts can be increased.

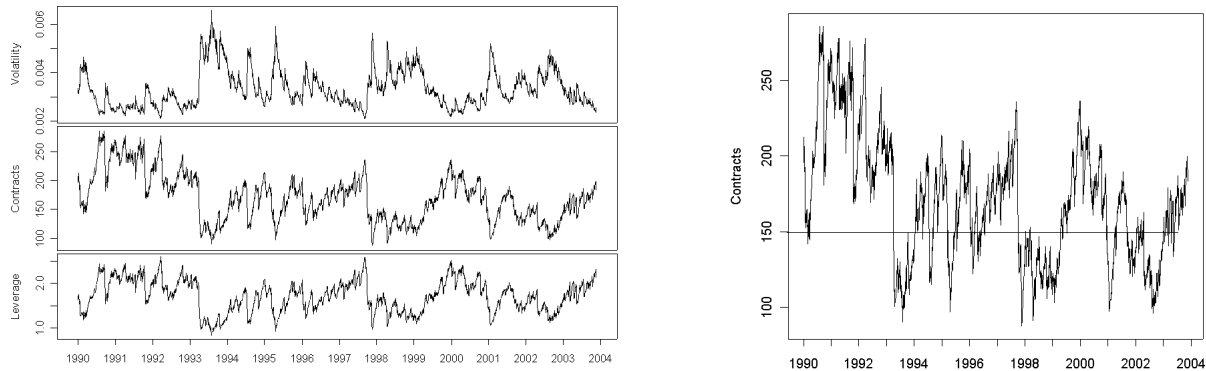


Figure 12: Time Varying Contracts to meet the 4% VaR target over one month with 95% confidence. This figure illustrates the inverse relationship between volatility and maximum number of contracts (or leverage). When the volatility is high the number of contracts shall be reduced. Conversely, when the volatility is low, the number of contracts can be increased. The horizontal line on the right is the maximum number of contracts for the static case (unconditional volatility).

6 Backtesting the VaR Models

The most straightforward way to backtest a VaR model is to plot daily P&L against predicted VaR, as recommended by RiskMetrics (see figure 13 from RiskMetrics).

Suppose we are testing a 95% VaR on 200 trading days, then the number of expected exceptions is $10 = 0.05 * 200$

If the number of exceptions is significantly higher than the expected value, the VaR is under estimated, and conversely too few violations indicate that the VaR is over estimated.

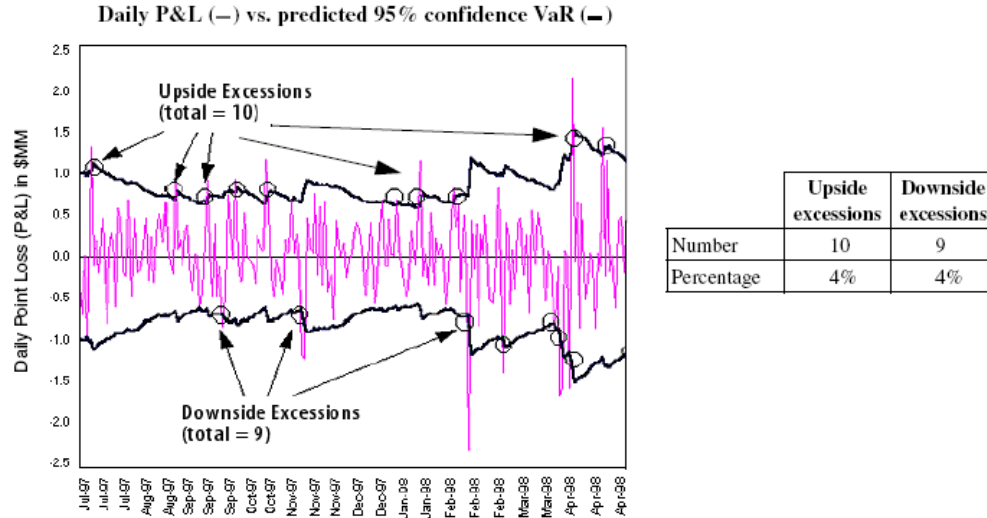


Figure 13: Exemple of Daily P&L vs Predicted 95% VaR source: RiskMetrics

Suppose that we are testing a VaR_p model with a critical level p (e.g. $p = 0.05$). For each day in history of T days, we determine whether or not an exception occurred. Let N be the number of exceptions. N/T should be as close as possible to the probability p . More precisely, N is distributed according a binomial distribution:

$$Prob(N) = \binom{T}{N} p^N (1-p)^{T-N} \quad (5)$$

with mean $E(N) = pT$ and variance $V(N) = Tp(1-p)$. Note that the standard deviation of N may be large compared to the expectation. In many statistics textbooks normal distribution approximations are available for the binomial distribution. These are not to be used here, because they only apply for binomials where the failure probability is "not too extreme"; The normal approximation is only valid if $0.1 < p < 0.9$ and we are interested in the case where $p = 0.05$ or $p = 0.01$.

The interval for N usually is large. For example, if p is 1 percent and T is equal to 255, we accept the model as long as $N < 7$ at the test confidence level 95 percent. However, there is a high probability that $N < 7$ and the model is incorrect. To measure the decision error, classically type I and type II error are involved. Type I error is the probability that the model is correct, but we reject it, and type II error is the probability that the model is incorrect, but we accept it. A higher test confidence level leads to a smaller type I error but a larger type II error.

VaR Confidence Level	Nonrejection region for number of exceptions N		
	$T = 255$ days	$T = 510$ days	$T = 1000$ days
99%	$N < 7$	$1 < N < 11$	$4 < N < 17$
97.5%	$2 < N < 12$	$6 < N < 21$	$15 < N < 36$
95%	$6 < N < 21$	$16 < N < 36$	$37 < N < 65$
92.5%	$11 < N < 28$	$27 < N < 51$	$59 < N < 92$
90%	$16 < N < 36$	$38 < N < 65$	$81 < N < 120$

Table 8: Model back testing, 95% non rejection test confidence regions (Source: Jorion [9])

The table 8 is the Basel Committee rules for back testing with the test confidence level to be 95 percent. It provides a simple way to accept or reject the current VaR model.

Suppose there are two years of data ($T = 510$ days), then the expected number of exceptions is $510 * 0.05 = 26$. However, by using the table 8, VaR user will not reject the models as long as N in the range [16, 36]. Values N greater to 36 indicate that the VaR is too high and the VaR under estimate actual losses. Values of N lesser than 16 indicates that the VaR is over conservative.

The table 8 also shows that, as the sample size increases, the interval N/T shrinks, which means we can reject a model more easily if the sample size is large. For example, consider the row with a 95 percent confidence level. When $T = 255$, the N/T interval is approximately (0.024, 0.082); when $T = 510$, the interval becomes (0.031, 0.071); when $T = 1000$ days, the interval is (0.037, 0.065).

Therefore, backtesting the VaR at a daily basis and for large confidence levels may help, even if the actual VaR is required for longer time intervals and higher confidence (let say 10 days VaR at 99% such as the Basle rule).

As recommended in the Risk Management Guide of RiskMetrics [14] p. 54: "A 95% daily confidence level is practical for backtesting because we should observe roughly one excession a month (one in 20 trading days). A 95% VaR represents a realistic and observable adverse move. A higher confidence level, such as 99%, means that we would expect to observe an excession only once in 100 days, or roughly 2.5 times a year. Verifying higher confidence levels thus requires significantly more data and time. Even if VaR is based on a high confidence level, it may make sense to test at other confidence levels as well in order to dynamically verify model assumptions (e.g., test at 90%, 95%, 97.5%, 99%). An even better test would be to compare the actual distribution of returns against the predicted distribution of returns (i.e., how close is the picture of predicted risk to the actual risk)..."

The tables (9 and 10) describes the backtesting results over a period of more than 14

years. The test results are describe as a pvalue. A pvalue is the probability to make an error if we reject the model. In other word, we can accept the model with confidence levels smaller than the pvalue. For example, if the pvalue is 0.07, then we can accept the model at the critical level 0.1 (confidence level 90%), but we should reject the model if we consider a 0.05 critial level (95%).

The best model is the GARCH(1,1) model. The historical volatility and RiskMetrics VaR models are also quite acceptable. The RiskMetrics model still yields good results at a 0.01 VaR level. However, for extreme rare events ($VaR.p = 0.001$, e.i for events that occur once in 4 years) no volatility model is able to predict the VaR. In case of short positions, we examine the positive tails The results are better than for long position. It can be explained by the facts that positive tail is more "gaussian" than the negative tail (see 2).

As recommended in the Risk Management Guide of RiskMmetrics [14], we set a database of daily VaR, plot daily P&L against predicted VaR and to monitor the number of excessions, or departures, from the confidence bands. According to the BIS, we look at the number of excessions over the most recent 12 months of data (or 250 trading days) as the basis for supervisory response. Excessions should be within confidence level expectations.

In addition to counting the percentage of VaR band excessions, risk monitors should watch out for excession clustering. For example, even if a quarterly backtest shows exactly 5% upside and 5% downside excessions, it would be a disturbing sign if these excessions were clustered in one narrow time period. Clustering of excessions could imply high autocorrelation in risks, which may manifest itself as a losing streak. The autocorrelation of excessions on FGBL can be "viewed" on the figure 15. The autocorrelation and losing streak issues will be adressed in a follow on paper on drawdown measures.

7 Multi Day Prediction

While most financial firms use one day VaR analysis for internal risk assessment, regulators require VaR estimates for 10 day returns or 20 day returns. There are two ways to implement a multi day VaR. If the time horizon is denoted by T, one can either look at past non overlapping T day returns, and use these in the same fashion as the one day VaR analysis, or extrapolate the one day VaR returns to the T day VaR. The latter method has the advantage that the sample size remains as it is. Possibly for this reason, RiskMetrics implements the latter method by the so called "square root of time" rule which implies that returns are normal with no serial correlation. However, for fat tailed data, a $T^{1/\alpha}$ is appropriate, where α is the tail index (see 2).

It is not possible to backtest the $T = 20$ days VaR. estimates because we have to compare the VaR predictions with non overlapping T day returns. This implies that the sample

$VaR.p = 0.05$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	0.379	0.150	1.8e-07	0.00057	0.779
estimate	0.053	0.044	7.1e-02	0.06362	0.049
conf.int1	0.046	0.038	6.2e-02	0.05552	0.042
conf.int2	0.062	0.052	8.0e-02	0.07252	0.057
$VaP.p = 0.01$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	0.00010	0.5394	1.3e-26	6.1e-15	3.2e-05
estimate	0.01735	0.0110	3.3e-02	2.6e-02	1.8e-02
conf.int1	0.01317	0.0077	2.8e-02	2.1e-02	1.4e-02
conf.int2	0.02242	0.0151	4.0e-02	3.2e-02	2.3e-02
$VaR.p = 0.005$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	9.4e-06	0.2630	2.4e-30	3.4e-19	3.2e-08
estimate	1.1e-02	0.0064	2.5e-02	1.9e-02	1.3e-02
conf.int1	7.9e-03	0.0040	2.0e-02	1.5e-02	9.5e-03
conf.int2	1.5e-02	0.0098	3.1e-02	2.5e-02	1.8e-02
$VaR.p = 0.001$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	4.0e-07	7.8e-01	1.1e-31	3.2e-16	1.6e-13
estimate	4.9e-03	6.1e-04	1.3e-02	8.2e-03	7.3e-03
conf.int1	2.8e-03	7.4e-05	9.2e-03	5.4e-03	4.7e-03
conf.int2	7.9e-03	2.2e-03	1.7e-02	1.2e-02	1.1e-02

Table 9: VaR backtest for the FGBL contract. This table indicates the backtesting results over the period starting Nov 23, 1990, ending October 11, 2004. The pvalue is the result of the binomial test related to the equation 5. The "estimate" is simply the number of exceptions divided by the number of days. This estimate should be close to the target level $VaR.p$. The confidence interval at 95% is also given. To accept the model at 95% confidence, the target $VaR.p$ shall be included in this confidence interval [$conf.int1, conf.int2$]

available for testing is T times smaller than the one day sample. Since we are looking at uncommon events, we need to backtest over a large number of observations. For one day VaR, 250 days is a minimum test length (as required by Basle, for example). Therefore, for 20 days VaR we would need 5000 days in the test sample!

8 Model Risks

The different models lead to different values of VaR and CVaR that can be interpreted as a model risk. This risk is minimized by taking the worst case over all estimated risk measures

$VaR.p = 0.05$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	0.749	0.230	2.4e-11	3.8e-08	0.128
estimate	0.051	0.045	7.7e-02	7.2e-02	0.044
conf.int1	0.044	0.038	6.8e-02	6.4e-02	0.037
conf.int2	0.059	0.053	8.7e-02	8.2e-02	0.052
$VaR.p = 0.01$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	0.017	0.7252	3.7e-17	1.4e-11	0.6611
estimate	0.014	0.0091	2.8e-02	2.4e-02	0.0107
conf.int1	0.011	0.0062	2.2e-02	1.9e-02	0.0074
conf.int2	0.019	0.0130	3.4e-02	3.0e-02	0.0148
$VaR.p = 0.005$	Historical	RiskMetrics	RogerSatchell	Parkinson	GARCH(1,1)
p.value	0.00026	0.4588	1.4e-18	1.8e-11	0.1715
estimate	0.01005	0.0040	1.9e-02	1.5e-02	0.0067
conf.int1	0.00692	0.0021	1.5e-02	1.1e-02	0.0042
conf.int2	0.01408	0.0068	2.4e-02	2.0e-02	0.0101
$VaR.p = 0.001$	Historical	RiskMetrics	Roger Satchell	Parkinson	GARCH(1,1)
p.value	0.00004	0.0873	4.7e-21	4.0e-07	0.0021
estimate	0.00396	0.0000	9.7e-03	4.9e-03	0.0030
conf.int1	0.00211	0.0000	6.7e-03	2.8e-03	0.0015
conf.int2	0.00676	0.0011	1.4e-02	7.9e-03	0.0056

Table 10: VaR backtest for the FGBL contract (for short positions).

under the different models.

$$VaR = \min_{i=1, N(\text{models})} VaR_i$$

$$CVaR = \min_{i=1, N(\text{models})} CVaR_i$$

where VaR_i (resp $CVaR_i$) is the VaR (resp CVaR) estimated under model i such as the Normal (gaussian) model, Historical Simulation Cornish Fisher approximation Extreme Value based estimation of VaR and CVaR (Hill estimation, Generalized Pareto Distribution fitting), GARCH model, ...

Conversely, the number of contracts is derived as the minimum number of contracts for all VaR and equivalent CVaR. Note that the actual number of contracts is dominated by the CVaR equivalent models rather than pure VaR (this is due to fat tails).

From the above result, from all the static VaR and CVaR estimates we are able to derive the maximum number of contracts to comply with a 4% one month VaR at 95% confidence

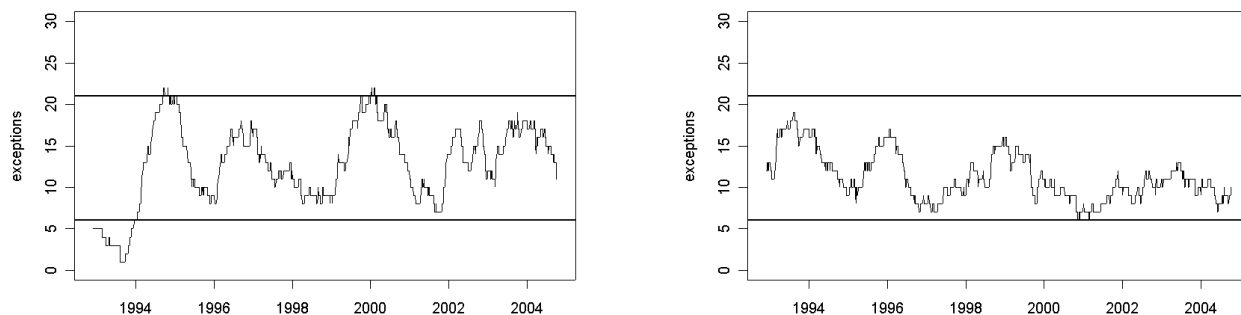


Figure 14: Number of exceptions for a 95% VaR. The number the VaR is exceeded is computed over a rolling window of 255 days (one year lag). The expected number of exceptions should be in the range 6 to 21 (confidence interval at 95%). The VaR is based on a GARCH(1,1) volatility model. The model can be accepted as the number of exceptions stay within the confidence band. The left graphics is for long positions, the right one is for short positions. The short/right figure looks much more stable than the long/left figure. We can explain this fact by a thicker tail of negative returns.

level or the equivalent 5% CVaR. From the table 3.4.2, the minimum number of contracts is 141. It represents the maximum number of contracts to be held either long or short.

For the dynamic VaR, the same approach applies. The number of contracts will be derived from worst case of the selected volatility models (RiskMetrics and GARCH(1,1)).

9 Conclusion

In this first paper, we have studied some statistical properties of the Euro Bund Future contracts using more than 14 years historical daily quotes. Even if the Bund Future market looks less risky than other markets, we have found significant fat tails that will lead to underestimate actual risks under normal hypothesis. Tail index is in the range of 3-6 that is consistent with previous studies. We have estimated the Value at Risk and Conditional VaR for the Fund trading the FGBL. At 95% confidence level with an objective of a one month VaR not exceeding 4% of the fund asset value under management, the historical VaR tends to underestimate the actual losses. It seems preferable to consider the equivalent 5% CVaR objective that will lead to more consistent risk measure and less risky positions: 141 contracts rather than 160 contracts per 10,000,000 euros of the funds capital.

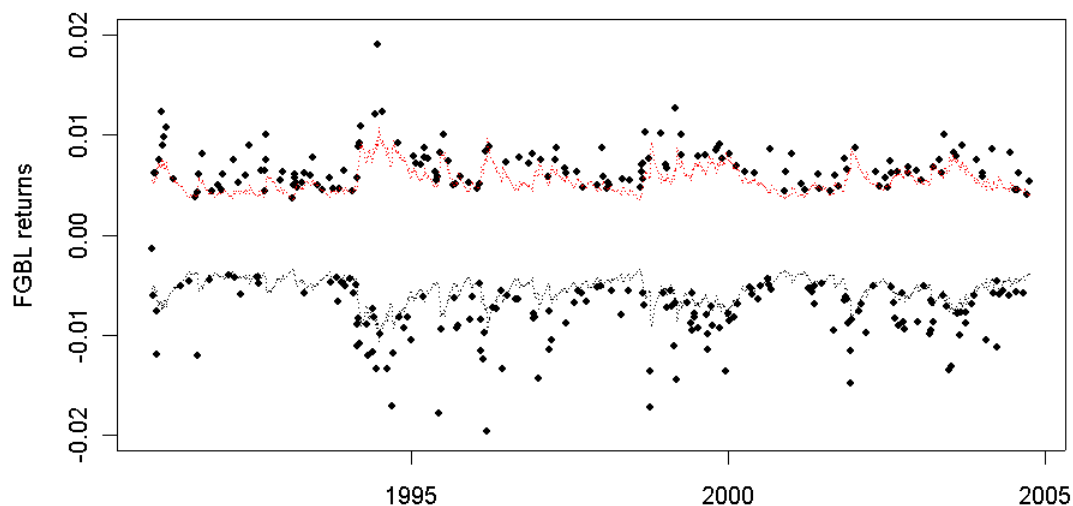


Figure 15: FGBL 95% VaR Excessions

Statistical estimation of risk and portfolio optimization are two important issues in risk management. In this paper, we have addressed the statistical modeling and VaR estimation. Portfolio optimization addresses the issue to find the best leverage under the different risk management constraints (VaR or CVaR, maximum drawdown, ...). Optimization will also take into account the "option like" remuneration structure of managers, such as management and incentive fees (high water mark) that may have major implications in terms of trader behavior and risk management.

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