1 Introduction

This document mainly addresses related distributions of a stochastic process, like the absolute return, the range, the maximum, the minimum, the first passage time, .... Such derived distributions are important to study, for many reasons:

- Pricing of exotic options.
- They can provide robust estimators, like volatility estimation via the high, low, or the range.
– derived distribution can be used in day to day trading, to perform estimation, like the expected time to execute of a limit order, or define a limit order with some target execution time, etc ...
– they allow to use more informative data on the data, that might be less noisy than usual end point samples.

Most of the results are developed in this section are given for a Brownian motion with drift.

However, we will address also the case of anomalous diffusions. Even if close form of distributions are not always possible, there may exists some strong properties that are still valid for more general processes.

For example, as far as the independence of increments is assumed and reflection properties apply (ie the process is starting afresh at any time see \[5\]). In any case, the Brownian motion case constitutes a good approximation for most cases.

Most of the extended formulas, including joint distributions of min, max, exit time from a corridor, etc ... are derived from the literature on exotic options (single or double barriers options), mainly \[1\] \[4\]

2 The maximum and the minimum of a Brownian motion

Consider a Brownian motion \(X(t)\) with constant drift \(\mu\) and diffusion \(\sigma^2\).

Notation :
– \(x_1 \leq 0\) will represent a lower barrier.
– \(x_2 \geq 0\) will represent an upper barrier.
– \(N(x)\) is the standard normal distribution function
– \(\phi(x) = dN/dx = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\) the standard normal density function.

The probability density function of an end point \(x\) and the maximum \(\leq x_2\) is :

\[
P(X(t) \in dx; \max \leq x_2; \mu, \sigma, t)/dx = \frac{1}{\sigma \sqrt{t}} \phi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) \left( 1 - e^{-\frac{4x_2^2 - 4x^2}{2sigma^2 t}} \right)
\]

for \(x \leq x_2\) and \(x_2 \geq 0\)

The probability distribution function of the running maximum is :

\[
P(\max \leq x_2; \mu, \sigma, t) = N \left( \frac{x_2 - \mu t}{\sigma \sqrt{t}} \right) - \exp \left( \frac{2 \mu x_2}{\sigma^2} \right) N \left( \frac{-x_2 - \mu t}{\sigma \sqrt{t}} \right) \quad \text{for } x_2 \geq 0
\]
The density function is obtained by derivation of the distribution with respect to $x_2$:

\[
P(max \in dx_2; \mu, \sigma, t)/dx_2 = \frac{1}{\sigma \sqrt{t}} \phi\left( \frac{x_2 - \mu t}{\sigma \sqrt{t}} \right) - 2\frac{\mu}{\sigma^2} \exp\left( \frac{2\mu x_2}{\sigma^2} \right) N\left( \frac{-x_2 - \mu t}{\sigma \sqrt{t}} \right) + \exp\left( \frac{2\mu x_2}{\sigma^2} \right) \frac{1}{\sigma \sqrt{t}} \phi\left( \frac{-x_2 - \mu t}{\sigma \sqrt{t}} \right)
\]

(3)

For a standard brownian motion, the maximum on interval $[0,1]$, the distribution and density of the maximum is:

\[
P(max < x; 0,1) = 2N(x) - 1, \quad x \geq 0
\]

\[
P(max \in dx; 0,1) = \frac{2}{\sqrt{2\pi}} e^{\frac{x^2}{2}}, \quad x \geq 0
\]

(4)

One can compute the moment generating function using the Laplace transform of a gaussian distribution on the half positive line using:

\[
erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du
\]

\[
= \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \right)
\]

(5)

The moment and cumulants generating function are:

\[
g(s) = E e^{sX}
\]

\[
g(s) = \int_0^\infty e^{sx} \frac{2}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx
\]

(6)

\[
\log g(s) = \frac{s^2}{2} + \log \left( 1 - erf\left( \frac{-s}{\sqrt{2}} \right) \right)
\]

(7)

Then, developing near zero to get first derivatives and cumulants:

\[
\log g(s) = + \frac{2}{\sqrt{\pi}} s + \frac{\pi - 2}{\pi} s^2 + \frac{2(4 - \pi)}{\pi \sqrt{\pi}} s^3 + \frac{8(\pi - 3)}{\pi^2} s^4 + \frac{8(\pi - 3)}{24} + O(s^5)
\]

(8)
yields the first moments of the maximum of a standard brownian motion:

\[
\text{mean} = \frac{2}{\sqrt{\pi}} \\
\text{variance} = \frac{\pi - 2}{\pi} \\
\text{standardDeviation} = \sqrt{\frac{\pi - 2}{\pi}} \\
\text{skewness} = \frac{\sqrt{2}(4 - \pi)}{(\pi - 2)^{3/2}} \\
\text{kurtosis} = \frac{8(\pi - 3)}{(\pi - 2)^2}
\] (9)

The moments of the maximum of a drift less brownian motion with scaling \(\sigma\) on interval \([0, t]\), is given by the laplace transform:

\[
\text{mean} = \sigma \sqrt{t} \frac{2}{\sqrt{\pi}} \\
\text{standardDeviation} = \sigma \sqrt{t} \sqrt{\frac{\pi - 2}{\pi}}
\] (10)

The skewness and kurtosis do not depend on scaling \(\sigma \sqrt{t}\).

Just for fun ... to compute \(\pi\) : estimate the mean of the maximum \(< \log(\text{high}/\text{open})>\) and the volatility \(\sigma\) and get \(\pi\) as

\[
\pi \approx \frac{2\sigma}{< \log(\text{high}/\text{open})>^2} \text{!!!}
\] (11)

The moments of maximum of a brownian motion with drift \(\mu\) are much more complex and closed form, still feasible, are not available easily.

This density function of running maximum is represented on the following figure, for different values of \(\mu\) and \(\sigma\).

For highest values of \(\mu\) in the order of magnitude of \(\sigma\), The distribution becomes more centered, closer to the dominant term, that is the distribution of the return itself. This is the case for long term investments.

For negative or small value of \(\mu\), the distribution is positively skewed, with most probable value being very closed to 0. This is the typical case of short term trading.

This difference between short and long term behavior is simply due to the difference in scaling factors for the mean scaling and the volatility : \(\mu(t)\) scales as \(t\) and \(\sigma(t)\) as \(\sqrt{t}\). Therefore the ratio \(\mu(t)/\sigma(t)\) as \(t/\sqrt{t} = \sqrt{t}\)
Consider for example an asset with 100% year drift and 100% volatility. Looking at the graph of such an asset it is very easy to see the drift ... Things are quite different at the daily level, the daily volatility is still rather high: 6.2%, while the daily drift is very small 0.38%, movements are becoming more and more dominated by noise, this is one of the difficulties of short term drift estimation.

The same considerations apply for the Minimum.

We can derive the probability functions for the minimum by changing $\mu$ to $-\mu$ and $x$ to $-x$, in the upper formulas, that is

\[ P(X(t) \in dx, \min < x_1; \mu, \sigma, t) = P(X(t) \in dx, \max \geq -x_1; -\mu, \sigma, t) \] \hspace{1cm} (12)

and

\[ P(\min < x_1; \mu, \sigma, t) = P(\max > -x_1; -\mu, \sigma, t) \] \hspace{1cm} (13)
If the probability for a single barrier are rather well known, this is not the case for the joint distribution of the Maximum and the Minimum: the Maximum and the Minimum are not independent variables, they are correlated.

The joint density of the Minimum and the Maximum and the end point \( x \) is:

\[
P(X(t) \in dx, \text{Minimum}(t) > x_1, \text{Maximum}(t) < x_2)/dx = \exp \left( \frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2} \right) \sum_{n=\infty}^{+\infty} \frac{1}{\sigma \sqrt{t}} \left[ \phi \left( \frac{x - 2n(x_2 - x_1)}{\sigma \sqrt{t}} \right) - \phi \left( \frac{x - 2n(x_2 - x_1) - 2x_1}{\sigma \sqrt{t}} \right) \right] \tag{14}
\]

for \( x_1 < 0 \) and \( x_2 > 0 \)

The joint probability distribution function of the minimum and the maximum is:

\[
P(\text{Minimum} \geq x_1, \text{Maximum} \leq x_2; \mu, \sigma, t) = \sum_{n=-\infty}^{+\infty} \exp \left( \frac{2n \mu(x_2 - x_1)}{\sigma^2} \right)
\]

with

\[
A = N \left( \frac{x_2 - \mu t - 2n(x_2 - x_1)}{\sigma \sqrt{t}} \right) - N \left( \frac{x_1 - \mu t - 2n(x_2 - x_1)}{\sigma \sqrt{t}} \right)
\]

\[
B = N \left( \frac{x_2 - \mu t - 2n(x_2 - x_1) - 2x_1}{\sigma \sqrt{t}} \right) - N \left( \frac{x_1 - \mu t - 2n(x_2 - x_1) - 2x_1}{\sigma \sqrt{t}} \right)
\]

for \( x_1 \leq 0 \) and \( x_2 \geq 0 \)

or in the equivalent form via the Fourier sine transform ( )

\[
P(\text{Minimum} \geq x_1, \text{Maximum} \leq x_2; \mu, \sigma, t) = \exp \left( -\frac{\mu^2 t}{2\sigma^2} \right) \sum_{n=1}^{+\infty} \frac{2n \pi}{n^2 \pi^2 + \left( \frac{\mu(x_2 - x_1)}{\sigma^2} \right)^2} e^{-\frac{n^2 \pi^2 (x_2 - x_1)^2}{2} \sin \left( \frac{n\pi (x_2 - x_1)}{x_2} \right)} \tag{15}
\]

Beyond its apparent complexity, series converge rapidly. The terms of the series decrease as \( e^{-cn^2} \) and, practically few terms are enough to get \( 10^{-6} \) precision.

The joint density is obtained

\[
P(\text{Minimum} \in dx_1, \text{Maximum} \in dx_2) = -\frac{\partial^2 P(\text{Minimum} \geq x_1, \text{Maximum} \leq x_2)}{\partial x_1 \partial x_2} \tag{16}
\]
This joint density function is sufficiently smooth and well-behaved to be used within a log likelihood maximizer \(^1\) that usually provide more accurate results than any other estimator.

The joint distribution of \((\max_{t \leq T} S_t, \min_{t \leq T} S_t, S_T)\) for a Brownian motion with \(\sigma = 1\) and drift \(\lambda\) is (see \([1]\])

\[
P \left( \max_{t \leq T} S_t \in dx_2, \min_{t \leq T} S_t \in dx_1, S_T \in dx \right) = I_{x_1 < x < x_2} e^{-\frac{x^2}{2} + \lambda x} \sum_{n=-\infty}^{+\infty} 4n \left( (n+1) \phi''_T(2x - x + 2n\delta) - n\phi''_T(x + 2n\delta) \right)
\]

(17)

where \(\delta = x_2 - x_1\) and

\[
\phi''_T(x) = \frac{d^2}{dx^2} \phi_T(x) = \frac{x^2 - T}{\sqrt{2\pi T^3}} \exp \left( -\frac{x^2}{2T} \right)
\]

(18)

(can also be derived from ??)

Given a time series \(S_{\text{open}}(t_i), S_{\text{high}}(t_i), S_{\text{low}}(t_i), S_{\text{close}}(t_i)\) for \(i = 1, \ldots N\), one computes

\[
x_{2i} = \log \frac{S_{\text{high}}(t_i)}{S_{\text{open}}(t_i)}, \quad x_{1i} = \log \frac{S_{\text{low}}(t_i)}{S_{\text{open}}(t_i)}, \quad x_i = \log \frac{S_{\text{close}}(t_i)}{S_{\text{open}}(t_i)}
\]

(19)

and find the maximum likelihood values for \(\mu\) and \(\sigma\) such that the PDF of \((x_{2i}/\sigma, x_{1i}/\sigma, x_i/\sigma)\) be given by formula 1 with

\[
\lambda = \frac{\mu}{\sigma} - \frac{\sigma^2}{2}
\]

(20)

For \(\sigma\), the maximum likelihood based on any of those joint densities is very accurate and provide a very good volatility estimator. However, for \(\mu\) will not provide, the MLE estimator is disappointing. Maximizing the product of for the different points lead to the close form of

\[
\mu - \sigma^2/2 = \frac{\sum x_i}{NT}
\]

(21)

independently from the highs and the lows! Saying differently, the relative distribution of the highs and the lows with respect to opens and closes does not depend on the drift. The highs and the lows bring no information on the drift!!!

\(^1\) Accurate computation is particulary crucial in parameters fitting with log likelihood. the joint density function can be very small and required accurate evaluation of derivatives at second orders, using some maximizer like Broyden-Fletcher-Goldfarb-Shanno algorithms. YATS provides analytic infinite series operators to accurately evaluate such expression, as well as partial derivative rules and infinity series derivation / integration resulting in very fast and accurate determination of parameters using parametric maximum likelihood methods.

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3 Absolute return

The distribution of the absolute value of a drift less brownian motion is given by:

$$P(|X_t| \in dy) = \frac{2e^y}{\sigma \sqrt{t}} \varphi\left(\frac{e^y}{\sigma \sqrt{t}}\right) dy$$

(22)

where $\varphi$ is the standard normal density.

4 The range

5 First Passage Time

5.1 Single barrier.

The first passage time is the minimum time to hit a barrier. Without loss of generality we will suppose that $X(0) = 0$. Then the passage time is:

$$\tau_x = \inf\{t \geq 0 ; X(t) = x\}$$

(23)

The distribution of the first passage time for a Brownian motion with drift is very well known (Karatzas [5] p. 197). In case of drift less motion, the reflection principle applies and it is easy to show that:

$$P(\tau_x < T) = P(\max_{[0,T]} X_t > x) = 2P(X_T > x) = 2N\left(\frac{-x}{\sqrt{T}}\right)$$

(24)

This result should is very robust and holds as far as we can apply the reflecition principle:

- symmetrical process : i.e. a null drift : this is nearly the case in short term returns.
- ”starts afresh” at any time : consequence of the strong Markov property.

For a Brownian motion, the density function is:

$$P(\tau_x \in dt) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-x^2/2t} dt; \ t > 0$$

(25)

For a Brownian motion with drift, the First Passage Time is derived form a drift less process using the Girsanov theorem to $X_t = \mu t + B_t(0,1)$.

$$P^\mu\left(X_T \in dx \text{ and } \max_{[0,T]} X_t < x_2\right) = \frac{dx}{\sqrt{2\pi T}} \exp\left(-\frac{x^2}{2T}\right) - \exp\left(-\frac{(2x_2 - x)^2}{2T}\right)$$

(26)
hence

\[ P \left( X_T \in dx \text{ and } \max_{[0,T]} X_t < x_2 \right) = \psi_{x_2}^\mu(x,T)dx \quad (27) \]

with

\[ \psi_{x_2}^\mu(x,T) = \frac{1}{\sqrt{2\pi T}} \left( \exp \left( -\frac{(x - \mu T)^2}{2T} \right) - \exp \left( 2\mu x_2 - \frac{(2x_2 + \mu T - x)^2}{2T} \right) \right) \quad (28) \]

In particular,

\[ P \left( \max_{[0,T]} X_t < x_2 \right) = \int_{-\infty}^{x_2} \psi_{x_2}^\mu(x,T)dx \quad (29) \]

\[ = N \left( \frac{h}{\sqrt{T}} - \mu \sqrt{T} \right) - \exp(2\mu x_2)N \left( -\frac{x_2}{\sqrt{T}} - \mu \sqrt{T} \right) \quad (30) \]

For a Brownian process \( X(\mu, \sigma) \) The density of the First Passage Time is

\[ P(\tau \in dt) = \frac{|x_2|}{\sqrt{2\pi \sigma^2 t}} \exp \left( -\frac{(x_2 - \mu t)^2}{2\sigma^2 t} \right) dt; \quad t > 0 \quad (31) \]

The probability of hitting a barrier \( x_2 \geq 0 \) at time \( \tau \), before the time \( t \) is :

\[ P(\tau < t; x_2; \mu; \sigma) = N \left( \frac{-x_2 + \mu t}{\sigma \sqrt{t}} \right) - \exp \left( 2\mu x_2 / \sigma^2 \right) N \left( \frac{-x_2 - \mu t}{\sigma \sqrt{t}} \right) \quad (32) \]

Can be compared to probability of the maximum \((2))\) and one can check that :

\[ P(\tau \in dt) = -\frac{d}{dT}P \left( \max_{0,T} < x_2 \right) \quad (33) \]

The First Passage Time is Inverse Gaussian distributed \(^2\).

The probability functions can be defined for lower barriers, ie \( x_1 < 0 \), mirroring the process. The pdf (resp. cdf) of the passage time for a lower barrier \( x_1 < 0 \) and drift \( \mu \) is the pdf (resp. cdf) of the passage for the upper barrier \(-x_1 > 0 \) and drift \(-\mu \).

The following figure represents the first hitting time for a Brownian motion with \( \sigma = 0.5 \), a barrier \( x = 0.5 \), and varying drifts from -0.5 to +0.5.

\(^2\) Inverse Gaussian is characterized by 2 parameters : the mean \( m \) and the scale \( l \). The pdf of an inverse gaussian is \( f(x;m,l) = l \frac{x^{-1/2}}{2\pi e} \exp\left(-\frac{l}{2m^2 x} (x - m)^2 \right) \) for the passage time distribution, \( m = x_2 / \mu \) and \( l = x_2^2 \sigma^2 \) the variance is \( m / l \), skewness \( 3 \sqrt{m / l} \) and kurtosis \( 15 \frac{m}{l} \).

It is possible to demonstrate that The logarithmic investment returns are independent and normally distributed is equivalent to testing the hypothesis that the First Passage Time is Inverse Gaussian distributed.
By letting $t \to \infty$ in the distribution function,

$$P(\tau < \infty) = e^{\exp(\mu x_2 - |\mu x_2|)}$$

(34)

"In particular, a Brownian motion with drift $\mu \neq 0$ reaches the level $x_2$ with probability one if and only if $\mu$ and $x_2$ have the same sign. If $\mu$ and $x_2$ have opposite signs, the density is defective, in the sense that $P(\tau < \infty) < 1$" (quoted from [5]).

Therefore, for negative drift (including drift 0), the distribution is quite horrible .... All moments, (the mean, the standard deviation) are $\infty$.

$$E(\text{passage Time}) = +\infty \quad \text{if } \mu \leq 0$$

For positive drift, the density is less wild but still positively skewed and fat tailed : very long waiting time are not so rare ....

The expected passage time for a positive barrier $x_2 > 0$ is :

$$E(\text{passage Time}) = \frac{x_2}{\mu} \quad \text{if } \mu > 0$$
The expected passage time does not depend on the volatility $\sigma$. This result is rather counter intuitive.

The other moments of the first passage time are

3. \[
\text{variance} = \sigma^2 \frac{x_2}{\mu^3}
\]

3. \[
\text{skewness} = 3 \frac{\sigma}{\sqrt{\mu x_2}}
\]

3. \[
\text{kurtosis} = 15 \frac{\sigma^2}{\mu x_2}
\]

All moments varies in the same direction of $\sigma$, this is easily understandable. It more interesting to notice that moments vary in inverse direction of $\mu$. This illustrate the sensitiveness and dispersion of the passage time with small $\mu$, as it is usually the case. Note also that the standard deviation can be reduced with smaller barriers. But decreasing the barriers will increase the fatness of the passage time, with higher kurtosis.

The density function is raising quite quickly to its maximum and decay slowly from this maximum to greater waiting times. The maximum of the density function, also known as the most probable value or the mode is :

\[
\tau_{\text{mode}} = \frac{x_2}{3\sigma^2}
\]

whatever the drift, even for negative drift !!! An other counter intuitive. It can be interpreted as follows. In case of negative drift, the the best opportunity to hit the barrier is at the very beginning, due to the volatility. As the time is running, the process will certainly go far away from the the positive territory, with smaller and smaller probability to revert back up to the barrier.

This result has practical usage : when a sending a differed order, whatever it is, a limit buy order or stop-loss order, then the most probable waiting time is given by (39)

\[
\tau_{\text{most Probable}} = \frac{x_2}{3\sigma^2}
\]

---

\[3\] Moments can be retrieved from the cumulant generating function of the first passage time

\[
\kappa(r) = \left( \frac{\mu}{\sigma} - \sqrt{\frac{\mu^2}{\sigma^2} - 2r} \right) \frac{h}{\sigma}
\]
whatever the drift, even for negative drift!!! whatever the trend. Consider a stock with daily volatility of 0.02 (yearly volatility about 32%), and set an order at 3% from the entry price, then the most probable waiting time will be 

\[ \tau_{\text{mode}} = \frac{0.03}{3\sqrt{0.02}} = 0.75 \]

that is 3/4 trading day = about 6 hours. If order has not been cleared at the end of the day, then you may enter a very long waiting time and it would be probably better to cancel it.

In [3], the question of first passage time is linked to the "Optimal Investment Horizons", defined as the minimum time to reach a given level and to propose this so called "optimal investment time" as the most probable value of the first passage time. It would be more correct to say that the waiting time will probably be greater than this most probable value.

The distribution of hitting time shows that it would be very dangerous to build a trading system on a single barrier. By defining upper and lower limits we expect to shorten the time spent in a trade.

Note that all results are given for log normal hypothesis. Under actual market price the distribution seems to be different. In [8], Baviera uses this discrepancy argument to reject the efficiency of market. However, it seems quite difficult to reject the inverse gaussian of the passage time, due to very large intrinsic errors of the model. Goodness of fit for inverse gaussian should be developed for financial market [7].

5.2 Exit time from a double barriers.

As for single barriers, the different probability functions for exit time are still related to the mirror principle and the probability functions for the maximum and the minimum. However relations are much more complex, and the iterated mirror principle shall apply [1].

We note \( \tau_{x_1,x_2} \) the first time the process (corridor) hits one of the two barriers \( x_1 < 0 \) or \( x_2 > 0 \). \( \tau_{x_1,x_2} \) relates to the simple barriers hitting time as follow :

\[
P(\tau_{x_1,x_2} \geq t; x_1, x_2) = P(\inf(\tau_{x_1}, \tau_{x_2}) \geq t) = P(\tau_{x_1} \geq t, \tau_{x_2} \geq t)
\]  

(41)

The expected exit time that is

\[
T(x_2, x_1) = E(\tau_{x_1,x_2})
\]  

(42)

Let \( p \) the probability that \( x_2 \) is hit before \( x_1 \)

\[
p = P(\tau^{x_2}_x < \tau^{x_1}_x)
\]  

(43)

Consider a Brownian motion \( X_t = \mu t + \sigma B_t \).
Thanks to the fundamental theorem of martingales, one has:

\[ E(X_\tau) = \mu T(x_2, x_1) = x_2p + x_1(1 - p) \]  

(44)

then

\[ T(x_2, x_1) = \frac{x_2p + x_1(1 - p)}{\mu} \]  

(45)

The probability to hit \( x_2 \) before \( x_1 \) for a Brownian motion \( X_t = \mu t + \sigma B_t \) is

\[ p = \frac{e^{-2\frac{\mu}{\sigma^2} x_1} - 1}{e^{-2\frac{\mu}{\sigma^2} x_1} - e^{-2\frac{\mu}{\sigma^2} x_2}} \]  

(46)

Note that this probability is the same for a given \( \frac{\mu}{\sigma^2} \) ratio.

The expected exit time is

\[ T(x_2, x_1) = \frac{\left( e^{-2\frac{\mu}{\sigma^2} x_1} - 1 \right) x_2 + \left( 1 - e^{-2\frac{\mu}{\sigma^2} x_2} \right) x_1}{\mu \left( e^{-2\frac{\mu}{\sigma^2} x_1} - e^{-2\frac{\mu}{\sigma^2} x_2} \right)} \]  

(47)

Hereafter some details on the method, with quite interesting results on the path. Consider first a standard Brownian motion \( B_t \).

\[ \psi_{x_1, x_2, T}(x) = \frac{1}{dx} P(B_T \in dx \text{ and } \tau_{x_1, x_2} \geq T) \]  

(48)

\[ \psi_{x_1, x_2, T}(x) = 1_{x_1, x_2} \sum_{n=-\infty}^{+\infty} \phi_T(x + 2n\delta) - \phi_T(2x_2 - x + 2n\delta) \]  

(49)

where \( \delta = x_2 - x_1 \) and \( \phi_T(x) \) is the centered Gaussian distribution density of standard deviation \( \sqrt{T} \):

\[ \phi_T(x) = \frac{1}{\sqrt{2\pi T}} exp\left(-\frac{x^2}{2T}\right) \]  

(50)

With a drift \( \mu \), and process \( X_t = \mu t + B_t \):

\[ \psi_{x_1, x_2, T}^{\mu}(x) = e^{-\frac{\mu^2}{2} + \mu x} \psi_{x_1, x_2, T}(x) \]  

(51)

We will note \( \psi(x) = \psi_{x_1, x_2, T}^{\mu}(x) \). \( \psi \) verify the Kolmogorov equation:

\[ \frac{d}{dt} \psi = \frac{1}{2} \frac{d^2}{dx^2} \psi - \mu \frac{d}{dx} \psi \]  

(52)
Let $\varphi_{x_1,x_2,+}^\mu(t)$ (resp. $\varphi_{x_1,x_2,-}^\mu(t)$) be the PDF of $\tau_{x_2}^\mu$ (resp. $\tau_{x_1}^\mu$) conditioned by $\tau_{x_2}^\mu < \tau_{x_1}^\mu$ (resp. $\tau_{x_1}^\mu < \tau_{x_2}^\mu$) that is

$$\varphi_{x_1,x_2,+}^\mu(t) = \frac{1}{dt} P(\tau_{x_2}^\mu \in dt \text{ and } \tau_{x_2}^\mu < \tau_{x_1}^\mu)$$

(53)

$$\varphi_{x_1,x_2,-}^\mu(t) = \frac{1}{dt} P(\tau_{x_1}^\mu \in dt \text{ and } \tau_{x_1}^\mu < \tau_{x_2}^\mu)$$

(54)

It can be shown that (see [1]) :

$$\varphi_{x_1,x_2,+}^\mu(t) = -\frac{1}{2} \frac{d\psi}{dx}(x_2)$$

(55)

$$\varphi_{x_1,x_2,-}^\mu(t) = \frac{1}{2} \frac{d\psi}{dx}(x_1)$$

(56)

with

$$\frac{d\psi}{dx}(x) = -\frac{2}{t} e^{-\frac{\mu x_1}{2} + \mu x} \sum_{n=-\infty}^{\infty} (x + 2n\delta \phi_t(x + 2n\delta))$$

(57)

where $\delta = x_2 - x_1$ and $\phi_t(x)$ is the centered Gaussian distribution density of standard deviation $\sqrt{t}$

The moment generating function of $\varphi_{x_1,x_2,+}$ and $\varphi_{x_1,x_2,-}$ are $^4$:

$$E(e^{-rt_{x_2}^\lambda} | \tau_{x_2}^\lambda) = -\frac{e^{\lambda x_2} \sinh \sqrt{\lambda^2 + 2r(x_2 - x_1)}}{p \sinh \left( \sqrt{\lambda^2 + 2r(x_2 - x_1)} \right)}$$

$$E(e^{-rt_{x_1}^\lambda} | \tau_{x_2}^\lambda) = \frac{e^{\lambda x_1} \sinh \sqrt{\lambda^2 + 2r(x_2 - x_1)}}{(1 - p) \sinh \left( \sqrt{\lambda^2 + 2r(x_2 - x_1)} \right)}$$

(58)

Consider a Brownian motion, it can be shown that :

$$p = \frac{e^{-2\mu x_1} - 1}{e^{-2\mu x_1} - e^{-2\mu x_2}}$$

(59)

Consequently the expected exit time is

$$T(x_2, x_1) = \frac{(e^{-2\mu x_1} - 1) x_2 + (1 - e^{-2\mu x_2}) x_1}{\mu (e^{-2\mu x_1} - e^{-2\mu x_2})}$$

(60)

$^4$ moments will be retrieved via the characteristic function = log(laplace) at origin for the different derivatives, in practice one can use a Taylor development near 0
if $\mu \to 0$, one recover the well known formulas:

For a drift less Brownian process the expected exit time from the corridor $x_1, x_2$ is very simple:

$$T(x_1, x_2; \mu = 0) = -x_2 x_1$$

(61)

as well as the the probability to hit the barrier $x_2 > 0$ before the barrier $x_1 < 0$ for a drift less Brownian motion is given by (see [5])

$$P(x_2 \text{ before } x_1; \mu = 0) = \frac{-x_1}{x_2 - x_1}$$

(62)

In the general case of drifted process, we have here powerful tools to estimate different parameters of process through passage time.

For example, by estimating $p$ and $T(x_2, x_1)$ for 2 arbitrary barriers $x_2, x_1$, we can recover $\mu$ via

$$\mu = \frac{x_2 p + x_1 (1 - p)}{T(x_2, x_1)}$$

(63)

This result hold as a consequence of the martingale property and should be robust. Considering a trading system where, the trader win $x_2$ (limit target) and loose $x_1$ (stop loss), then $x_2 p + x_1 (1 - p)$ is simply the expected profit and $T(x_2, x_1)$ the expected time of the trade. In other word, we can estimate $\mu$ by recording the profits and time per trade!!!

$$\mu \simeq \frac{<\text{tradeProfit}>}{<\text{tradeTime}>}$$

(64)

it is even not required to get the expected time if one want to estimate the sign of $\mu$, the average of profits is enough. and show also that a winning system requires $\mu > 0$.

However the density of the exit time should provide a more precise and consistent estimator (see drift estimation methods).

Let define $\varphi_{x_1, x_2}^\mu$ the density of the exit time from the corridor

$$\varphi_{x_1, x_2}^\mu(t) = \frac{d}{dt} P(\tau_{x_1, x_2}^\mu < t)$$

(65)

can be obtained by getting the Laplace transform first:

$$E(e^{-rt\tau_{x_1, x_2}^\mu}) = \int_0^{+\infty} e^{-rt} \varphi_{x_1, x_2}^\mu(t) dt$$

(66)

$$E(e^{-rt\tau_{x_1, x_2}}) = pE(e^{-rt\tau_{x_1}} | \tau_{x_1}^\lambda < \tau_{x_2}^\lambda) + (1 - p)E(e^{-rt\tau_{x_2}} | \tau_{x_1}^\lambda < \tau_{x_2}^\lambda)$$

$$= \frac{e^{\mu x_1} \sinh(\sqrt{\mu^2 + 2r x_2}) - e^{\mu x_2} \sinh(\sqrt{\mu^2 + 2r x_1})}{\sinh(\sqrt{\mu^2 + 2r \delta})}$$

(67)
For a drift less process, one can derive easily the first moments, by developing the logarithm expression near 0,

\[
E(e^{-rτ_{x_1,x_2}}; µ = 0; σ = 1) = \frac{\sinh(\sqrt{2r} x_2) - \sinh(\sqrt{2r} x_1)}{\sinh(\sqrt{2r} δ)}
\]

\[
\log E(e^{-rτ_{x_1,x_2}}) \approx x_1 x_2 r - \left(x_1 x_2^3 + x_1^3 x_2\right) \frac{r^2}{6} + \left(3x_1 x_2^5 + 10x_1^3 x_2^3 + 3x_1^5 x_2\right) \frac{r^3}{90} + O(68)
\]

and get the first cumulants:

\[
E(τ_{x_1,x_2}; µ=0) = κ_1 = -x_1 x_2
\]

\[
\text{Variance}(τ_{x_1,x_2}; µ=0) = κ_2 = -\frac{1}{3} x_1 x_2(x_1^2 + x_2^2)
\]

\[
E((τ - \text{mean})^3) = µ_3 = κ_3 = -\frac{1}{15} x_1 x_2 \left(3x_2^4 + 10x_1^2 x_2^2 + 3x_1^4\right)
\]

One recognizes the well known result $-hl$ stated before. The distribution is strongly skewed with To get the results for a Brownian motion with scaling $σ$, replace $x_i$ by $x_i/σ$.

For the case of symmetric barriers, ie when

\[
Δ = x_2 = -x_1
\]

The Laplace function becomes:

\[
E(e^{-rτ_{Δ,Δ}}) = \frac{\cosh(µΔ)}{\cosh(\sqrt{µ^2 + 2r} Δ)}
\]

and first moments:

\[
E(τ_{Δ,Δ}) = \frac{Δ}{µ} \tanh(µΔ)
\]

\[
\text{Variance}(τ_{Δ,Δ}) = \left(\frac{\sinh(2µΔ)}{2µΔ} - 1\right) \frac{Δ^2}{µ^2 \cosh^2(µΔ)}
\]

\[
E(τ_{Δ,Δ}) \approx Δ^2(1 - \frac{1}{3} µ^2 Δ)
\]

\[
\text{Variance}(τ_{Δ,Δ}) \approx 2/3Δ^4
\]

and with $σ$:

\[
E(τ_{Δ,Δ}) \approx \frac{Δ^2}{σ^2} (1 - \frac{µ^2 Δ}{3σ})
\]

\[
\text{Variance}(τ_{Δ,Δ}) \approx \frac{2Δ^4}{3σ^4}
\]
\[ \sigma^2 \approx \frac{\Delta^2}{E(\tau_{\Delta,\Delta})} \] (76)

The density will be recovered via inverse transform of \( \tau \) (see [1] for details) (recall also that the Laplace transform can be used to get the moments by taking successive derivatives at 0). Finally the density of the first exit time from the double barriers \( x_1, x_2 \) for a Brownian motion with drift \( \mu \) is:

\[ \varphi^\mu_{x_1,x_2}(t) = \frac{d}{dt} P(\tau^\mu_{x_1,x_2} < t) \] (77)

\[ \varphi^\mu_{x_1,x_2}(t) = \frac{\pi}{\delta^2} e^{-\frac{\pi^2}{\delta^2}} \sum_{n=1}^{+\infty} (-1)^{n-1} n e^{-\frac{n^2 x_1^2}{\delta^2}} \left( e^{\mu x_1} \sin \frac{n\pi x_2}{\delta} - e^{\mu x_2} \sin \frac{n\pi x_1}{\delta} \right) \] (78)

This density can be used in a log maximum likelihood estimator.

For a process with diffusion \( \sigma \) it is quite easy to change the variable, using \( x/\sigma, \mu/\sigma, x_1/\sigma, x_2/\sigma \) (and adjusting by \( \sigma \) dimension in function as required). Example:

The density for the exit time is:

\[ \varphi^{\mu,\sigma}_{x_1,x_2}(t) = \sigma^2 \varphi^{\mu/\sigma}_{x_1/\sigma,x_2/\sigma}(t) \] (79)

### 5.3 First passage time for Anomalous Diffusion.

Fractional Brownian motion \( X(t) \) (see [6]) is a Gaussian process with \( X(0) = 0, <X(t)> = 0 \) and \( <[X(t) - X(s)]^2> = |t - s|^{2H} \) \((0 < H < 1)\) The exact First Passage Time distribution of this process is not known. It was conjectured (see [2]), based on scaling argument and numerical evidence, that for large \( t \), \( f(t) \) scales with \( t \) as

\[ f(t) \sim t^{H-2}. \] (80)

### 6 Applications

#### 6.1 How to estimate the limit and stop prices

Previous sections give powerful tools to accurately timing orders:
- estimating the time to close a position for a limit order,
- define a day stop-loss
- estimate the limit and stop-loss price for a given horizon.
By the way, this the objective of the Range Trading System: accurate definition of limit/stop loss orders at transaction costs and trading frequencies boundaries.

Many example of use have already been given in previous section. We can just recall some of them.

When a sending a differed order, whatever it is, a limit buy order or stop-loss order, then the most probable waiting time is given by (39) whatever the trend. Consider a stock with daily volatility of 0.02 (yearly volatility about 32%), and set an order at 3% from the entry price, then the most probable waiting time will be $\tau_{\text{mode}} = \frac{0.03^2}{3+0.02^2} = 0.75$ that is 3/4 trading day = about 6 hours. If order has not been cleared at the end of the day, then you may enter a very long waiting time and it would be probably better to cancel it.

In any case, the distribution of hitting time shows that it would be very dangerous to build a trading system on a single barrier. By defining the right upper and lower limits, we will adjust the time spent in a trade to operational targets.

7 Annex

7.1

8 References


