A Range Trading System

Daniel Herlemont
email:dherlemont@yats.com  -  YATS Finances & Technologies  -  tel:+33 (0) 5 62 71 22 84

This trading system addresses some questions like:

- probability of hitting a given level at different time horizons
- how to set optimal limit / stop prices vs expected time to close a position.
- or alternatively: what are the optimal investments horizons with a target price.
- compute probability functions for the realization of the trade within a given time frame
- quantify the so called "volatility trading" systems
- How to catch the highs ?
- what are the optimal leverage
- taking into account, sustainable trading frequencies, transaction costs, etc ...

The range trading system is very simple and is defined as follows, Consider \( S_t \) the prices of a risky asset, and a starting wealth \( W_{t_0} \)
Define also \( f \) as being a constant ratio of wealth invested in the risky asset.

1. At time \( \tau_i \), enter a long position at current market price: \( entryPrice = S_{\tau_i} \) i.e, invest \( fW_{\tau_i} \) \(^1\) in the risky asset, \( (1-f)W_{\tau_i} \) in cash or borrowed if \( f > 1 \) \(^2\)

2. Set a limit sell order at \( u\% \) return upper the entry price. The limit sell price is defined as \( \log(limitPrice/entryPrice) = u \) \( limitPrice = \exp(u)entryPrice \)

\(^1\) we will use also the notation \( W_i = W_{\tau_i} \)

\(^2\) (we will first suppose that the current risk free rates are zero, it is always)
3. Set a “stop loss” order at \(-d\%\) return below the entry price, \(\log(\text{stopPrice}/\text{entryPrice}) = -d \quad \text{stopPrice} = \exp(-d)\text{entryPrice}\)

4. The position is cleared as soon as the current price hits one of the two sell prices, yielding:
   
   - a winning trade if the price hits the upper limit price before the lower stop price: the gain is about \(u\%\) of the amount invested in the risky asset. More precisely, the new wealth balance is:
     
     \[ W_{i+1} = W_i + f(e^u - 1)W_i \]
     \[ \approx W_i(1 + fu) \quad \text{for} \quad u \ll 1 \]  
     
     A winning trade has a probability \(p\), being the probability of hitting \(u\%\) before \(-d\%\)
   
   - a losing trade if the if the price hits the lower stop price before the upper limit price. The loss is about \(-d\%\) of the amount invested in the risky asset. More precisely, the new wealth balance is:
     
     \[ W_{i+1} = W_i + f(e^{-d} - 1)W_i = W_i(1 - f(1 - e^{-d})) \]
     \[ \approx W_i(1 - fd) \quad \text{for} \quad d \ll 1 \]  
     
     with the probability \(q = 1 - p\), probability of hitting \(-d\%\) before \(u\%\)

5. Enter a new position, as soon as possible ... may depend on the current operational capabilities (for example, the trader may be constrained to wait until the next day or may not be available, or etc ... ). The constant ratio strategy will require to invest \(fW_{i+1}\), so that it is possible to optimize the system by simply defining limit orders for the the number of shares that will adjust the wealth to the targeted ratio. Balance optimization will be important when we will address the question of transaction costs.

The wealth \(W_i\) follows a random multiplicative process:

\[ W_{i+1} = W_i(1 + f(e^r_i - 1)) \]  

With \(r_i\) being the outcome of the trade \(i\), i.e. \(r_i = u\) with probability \(p\) or \(r_i = -d\) with probability \(q = 1 - p\)

Then, \(\log(W_n/W_0)\) is a binominal random variables, with following with mean 
\[ nE[\log(1 + f(e^r_i - 1))] = n\left(p\log(1 + f(e^u - 1)) + q\log(1 + f(e^{-d} - 1))\right) \]  
and variance \(npq(e^u - e^{-d})\)

Objective is to find \(f, u\) and \(d\), to maximize some utility function of the wealth for a given trading horizon \(T\).

We will first examine the system in the limiting approximations
maximize the logarithm of the wealth as the utility function. The logarithm utility is know as being the best utility with reinvestments.

stationary and log normal random walk for price distribution

$u$ and $d$ are small enough to apply classical approximations like $e^r - 1 \approx r$

without risk free

single asset

Time horizon and/or large number of trades

No transaction cost or operational constraints (like trading frequencies, settlements, slippage etc ...)

Then we will relax on those assumptions, one by one, and combine altogether to provide concrete solution in full operational market conditions and traders environments: ie:

maximize arbitrary utility function, including consumption.

different processes for the price models, spanning mean reverting models, stochastic volatility models, non parametric models,

arbitrary $u$ and $d$

short term horizon and limited number of trades

large deviation of prices and large fluctuations of wealth,

with risk free

multiple assets

with transaction cost and operational constraints (like trading frequencies, settlements, slippage etc ...)

This system can be viewed as an attempt to "reduce" the varying returns of the random prices into predefined outcomes ($u$ and $d$). On the other hand, the holding time of the position will fluctuate as a random process, know as "occupation time" or "first hitting time". This trading system is built on the duality that exists between time and space for any given stochastic process. This can be though as kind the Heisenberg principle: we can trade at fixed time without controlling the returns, or trade fixed target without controlling
the trading time. Considering the time as a random variable has been addressed previously in [2].

Technics for assessment of such trading System will share tools with exotic options pricing, one of the main challenge will be to have right models for the underlying, including first moments (mean and variance), distributions of highs, lows, first passage times, extreme events, ....

Probability of win vs loss Suppose that we are trading in stationary conditions, so that the probability of a winning trade, $p$, is constant. $p$ it is the probability that the level $u$ is reached before $-d$.

\[
p = \text{probabilityOfWinningTrade} = P(u \text{ before } -d)
\]
and the probability of a losing trade is

\[
q = \text{probabilityOfLosingTrade} = P(-d \text{ before } -u) = 1 - p
\]

The distribution of trading times is defined by the minimum time that the price will hit the $u$ or $-d$ barrier. We will first examine the simpler case of a single barrier, that is the well known question of the ”first hitting time”.

The exit time of a winning trade is defined as :

\[
\tau_u = \inf(t > 0; \text{price}(t) > \text{limitPrice}, \text{price}(0) = \text{entryPrice})
\]
\[
= \inf(t > 0; r(t) > u, r(0) = 0)
\]

and the exit time of a losing trade :

\[
\tau_d = \inf(t > 0; \text{price}(t) < \text{limitPrice}, \text{price}(0) = \text{entryPrice})
\]
\[
= \inf(t > 0; r(t) < -d, r(0) = 0)
\]

Finally, the trade time is

\[
\tau = \inf(\tau_u, \tau_d)
\]

The different distributions of first passage time and exit time have been studied (see section ??).

Consider the price $S_t$ being a log normal process with drift $\mu$ and volatility $\sigma$.

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t
\]

$log S_t$ follows an arithmetic Brownian motion with drift $\lambda$. 

Copyright 2004, Daniel Herlemont, email:dherlemont@yats.com YATS, All rights reserved, tel:+33 (0) 5 62 71 22 84
\[ \lambda = \mu - \frac{\sigma^2}{2} \]
\[ d \log S_t = \lambda dt + \sigma dB_t \]  
(10)

Not that it is important here to consider \( \lambda \) and not \( \mu \).

Then, the probability of a winning trade is \( p \) is probability of hitting \( u \) before \(-d\) is given by ?? :
\[ p = \frac{\exp(\alpha d) - 1}{\exp(\alpha d) - \exp(-\alpha u)} \]  
(11)

\[ q = 1 - p = \frac{1 - \exp(-\alpha u)}{\exp(\alpha d) - \exp(-\alpha u)} \]  
(12)

with \( \lambda > 0 \) and
\[ \alpha = 2 \frac{\lambda}{\sigma^2} = 2 \frac{\mu}{\sigma^2} - 1 \]  
(13)

We may consider
\[ \alpha d \ll 1 \text{ and } \alpha u \ll 1 \]  
(14)

A second order expansion of \( p \) and \( q \) given by 11 and lead to :
\[ p \approx \frac{d}{d+u} \left( 1 + u \frac{\lambda}{\sigma^2} \right) \]
\[ q \approx \frac{u}{d+u} \left( 1 - d \frac{\lambda}{\sigma^2} \right) \]  
(15)

\( p \) (resp \( q \)) is composed of 2 terms :

- the drift less probability \( d/(d+u) \) (resp. \( u/(d+u) \)) of hitting \( u \) before \( d \) (resp \( d \) before \( u \))
- augmented (resp. reduced) by a smaller term of order of \( u \) (resp. \( d \)) This very small bias will be responsible for transforming the strategy in a winning game over a null game in the drift less.

The drift has only the second order effect \( ud \) on probabilities, with possibly very small departure from the drift less case. Consider the simple example:
\[ u = d = 0.02 = 2\% \]
\[ \frac{\lambda}{\sigma^2} = 0.5 \]  
(16)
Then

\[ p \approx .5 + 0.05 \]
\[ q \approx .5 - 0.05 \]  

(17)  
(18)

ie a very small differences with drift less process that may be hard to detect and to rely on for trading !!! But, this is quite normal, there is no magic, the edge cannot be huge, however optimized money management technics (like optimal growth) and timing may transform very small and constant edge into very large wealth.

The arithmetic average per trade is

\[ a = p(e^u - 1) - q(1 - e^{-d}) = \frac{(e^u - 1)(exp(\alpha d) - 1) - (1 - e^{-d})(1 - exp(-\alpha u))}{exp(\alpha d) - exp(-\alpha u)} \]  

(19)

If \( \alpha d \ll 1 \) and \( \alpha u \ll 1 \)

\[ a \approx ud\frac{\alpha + 1}{2} = ud\frac{\mu}{\sigma^2} \]  

(20)

This arithmetic average is of second order approximates, so the edge is small, but still positive.

We also recover \( \mu \) (and not \( \lambda = \mu - \sigma^2/2 \))

If \( \mu = 0 \), then \( a \) is null, and it not possible to implement a winning ”game” ...

In case of \( u = -d = \delta \), we get a simpler close form for any \( \delta \):

\[ a = 2\frac{\sinh(\beta \delta)\sinh(\delta/2)}{\cosh(\alpha \delta/2)} \]  

(21)

with \( \beta = \mu/\sigma^2 \) and \( \alpha = 2\lambda/\sigma^2 = 2\mu/\sigma^2 - 1 \)

Consider the case \( \mu > 0 \), then \( a \) is \( > 0 \), and it is possible to implement a winning strategy using and find an optimal leverage \( f^* \) (see section ??) :

\[ f^* = \frac{a}{(e^u - 1)(1 - e^{-d})} = \frac{p(e^u - 1) - q(1 - e^{-d})}{(e^u - 1)(1 - e^{-d})} \]  

(22)  
(23)
If $d \ll 1$ and $u \ll 1$, after some series expansion development, we recover the well known criteria:

$$f^* \approx \frac{\mu}{\sigma^2}$$

In case of $u = -d = \delta$, we get a simpler close form for $f^*$,

$$f^* = \frac{\sinh(\beta \delta)}{2 \cosh(\alpha \delta/2) \sinh(\delta/2)}$$

(25)

for any $\delta$, with $\beta = \mu/\sigma^2$ and $\alpha = 2\lambda/\sigma^2 = 2\mu/\sigma^2 - 1$

One can use classical formulas for hyperbolic. Note also that

$$(e^u - 1)(1 - e^{-d}) = e^{(u-d)/2} \sinh(u/2) \sinh(d/2)$$

(26)

For small $u$ and $d$, we find the same leverage as for continuous trading that requires permanent and continuous adjustments to maintain a constant $\mu/\sigma$ proportion of the stock held in the portfolio. This result is quite interesting, if $u$ and $d$ are small enough (as it is usually the case), then this strategy is approaching the continuous "perfect" one, without continuous trading requirements. However, to get this results we had to approximate and $d$ and $u$ shall be small enough. Nevertheless, since the gradient of $f^*$ is null at $u = d = 0$, then neighbor points lead to second order differences in $f^*$

The wealth $W_n$ is a binomial random multiplicative process, then $\log(W_n/W_0)$ is a sum of the i.i.d random variables $\log(1 + f(e^r - 1))$, and the Central Limit Theorem applies: with mean $E_n = nE[\log(1 + f(e^r - 1))] = n\left( p \log(1 + f(e^u - 1)) + q \log(1 + f(e^{-d} - 1)) \right)$ and variance $V_n = npq(e^u - e^{-d})$

$$P \left( \alpha_1 \leq \frac{\log(W_n/W_0) - E_n}{\sqrt{V_n}} \geq \alpha_2 \right) = N(\alpha_2) - N(\alpha_1)$$

(27)

In other word, $\log(W_n/W_0)/n$, representing the geometric mean per trade tends to a limit $E[\log(1 + f(e^r - 1))]$, that should be maximized, to realize the best strategy with reinvestments. This is the well known Kelly criteria,

Optimal constant leverage is given by :

$$f^* = \arg \max_f E[\log(1 + f(e^r - 1))]$$

$$= \arg \max_f p \log(1 + f(e^u - 1)) + q \log(1 + f(e^{-d} - 1))$$

(28)

For any other constant leverage, the geometric mean per trade is

$$g(f) = (1 + f(e^u - 1))^p (1 + f(e^{-d} - 1))^q$$

$$\log(g(f)) = p \log(1 + f(e^u - 1)) + q \log(1 + f(e^{-d} - 1))$$

(29)
For $\alpha d \ll 1$ and $\alpha u \ll 1$,

$$\log(g(f)) \approx fa$$

$$g(f) \approx e^{fa} \approx 1 + fa \quad (30)$$

For optimal $f$,

$$g^* = g(f^*) \approx 1 + ud\frac{\mu}{\sigma^2}$$

$$\log(g(f^*)) \approx ud\frac{\mu}{\sigma^2} \quad (31)$$

How long are the execution of trades?

Positions are held until the price hits one of its barrier. A very important question is about the time to hit the barriers, and how long the trader has to wait before entering a new trade.

Consider the random variable $\tau$, the duration of a trade. This is the time that the price remains within the $u\%$ and $-d\%$ interval. For a Brownian motion, the formula has been developed in ?? (the section on first hitting time give numerous results on those distribution, including a close form for the density of the trading time)

$$E(\tau) = \frac{(e^{2\alpha d} - 1) u - (1 - e^{-2\alpha u}) d}{\lambda (e^{2\alpha d} - e^{-2\lambda u})} \quad (32)$$

For $\alpha d \ll 1$ and $\alpha u \ll 1$, we recover the well known formula for drift less Brownian process:

$$E(\tau) \approx \frac{ud}{\sigma^2} = \frac{ud}{\sigma \sigma} \quad (33)$$

The expected time to exit from small barriers does not depend on the drift, leading to important result on the expected trade execution time:

- proportional to the limit $u$ and stop loss $-d$,
- inverse proportional to the variance, ie the square of volatility
- does not depend on the drift.

Inversing the sentence to get the same thing for the trading frequency $(1/E(\tau))$:

- inverse proportional to the limit $u$ and stop loss $-d$,
- proportional to the variance, ie the square of volatility
• does not depend on the drift.

Can be compared with the expected time to exit from a single barrier that heavily depends on the drift: \( E(\text{exitTime}) = \text{barrier}/\lambda \)

As an application, when defining range orders, the trader will get quite robust estimate of the waiting time, whatever the drift.

Consider again the example (16) with a daily volatility \( \sigma = 0.02 \) (32% annualized), and \( u = d = 2\% \), \( E(\tau) \approx 1\text{day} \) for \( \sigma = 0.04 \) (65% annualized) and \( u = d = 2\% \), \( E(\tau) \approx 1/4\text{day} \)

As a rule of thumb, setting the limit and stop at volatility will result in a an expected trading position of one time unit. This is true at every scaling, daily, weekly, monthly, ... Recall that we are still in the log normal setting and approximation conditions. First, the distribution of the hitting times are quite singular with peaked most probable value, and very fat tails. Second, approximation conditions may not apply (at large time scale for example). Third, actual price dynamics are not so ”normal” (they are not so far as everybody claimed). As a conclusion, the expected trading time will certainly be smaller compared to the log normal school case. In [1] and [2]) the expected exit time from a double barriers seems not to be consistent with the log normal case. As already noticed, this results have to be carefully revisited with usual statistics tool for hypothesis testing.

How many trades within a given time period ?

Consider the simple setting that we are continuously trading and enter a new position as soon as the previous position was closed. Knowing the expected execution time of a trade, it is quite easy to derive the expected number of trades \( n \) during a given time period \( T \).

\[
n = \frac{T}{E(\tau)} \text{ with probability one.} \tag{34}
\]

or under usual approximation (14)

\[
n \approx T \frac{\sigma^2}{ud} \tag{35}
\]

TERMINAL WEALTH

Considering \( n \) trades, the terminal wealth is a geometric series of \( g(f) \)

\[
W_T = G_T W_0 \\
G_T = g(f)^n \\
= g(f)^{T/E(\tau)} \tag{36}
\]

or under usual approximation (14)

\[
G_T \approx e^{\mu T} \tag{37}
\]
This formula can also be retrieved via the Central Limit Theorem applied to the binomial multiplicative process $W_T$, where $\log W_T$ is approximatively gaussian distributed with mean $f \mu T$ and variance $f^2 \sigma^2 T$.

For optimal $f^*$

\[
G_T^* \approx e^{\frac{\mu}{\sigma^2} T} = e^{\frac{u^2}{\sigma^2}} = e^{2T\text{sharpe}^2}
\]

\[
W_T \approx e^{2T\text{sharpe}^2}W_0
\]

(39)

and quite surprisingly, the terminal wealth return does not depend neither on $u$ or $d$ !!!

This result is due to the fact that optimal trading is obtained for the limiting case $u = d = 0$, where the geometric mean is at its maximum, and therefore the gradient is null:

\[
\frac{\partial g(f)}{\partial u}(u = 0, d = 0) = 0
\]

\[
\frac{\partial g(f)}{\partial d}(u = 0, d = 0) = 0
\]

(40)

Therefore For $u$, $d$ near the optimal solution, ie, near 0, the variation of the geometric mean is of second order with $u$ and $d$, and will not depend so much on small variations of $u$ and $d$.

Operational constraints

In this first approach, we made a lot of hypothesis and approximations without considering operational constraints of number of trades per day, transactions costs, etc ... The main results are obtained with the hypothesis that the trader is able to enter a new trade as soon as the previous position is closed : in practice this is not realistic.

Look for example at the expected time of a trade : $E(\tau) = ud/\sigma^2$ and considering typical values, $E(\tau)$ can be very small. Consider for example the case of a typical case of a 5% daily volatility , and $u$ and $d$, few percents let say 1%. The expected time of the position is about the required trading rate is $0.01 * 0.01/0.05^2 = 0.04$ of a trading day, that is one trade every 20 minute in average. It is clear that this trading rate is not sustainable in actual operational conditions, if we consider also that this number is just an average value, and much smaller and longer will be not rare. ... The consequence of not trading at the required frequency will be missing trades in the overall wealth at the end of the trading period : $n_{\text{actual}} \ll n_{\text{theoric}}$, and the losses compared to the theorical wealth is exponential: $g^n_{\text{actual}} \ll G^n_{\text{theoric}}$ This remark illustrate the importance of trading at approaching optimized value as close as we can.

In actual condition, traders face stringent operational constraints that will limit the number of trades and cannot go beyond a certain number of trades per unit of time. Consider
the number of trades per unit time \( k \), then the number of trades during a period \( T \) is \( n = kT \), the expected number of trades per unit time \( k = 1/E(\tau) \) and \( k = \sigma^2/\ud \) and define \( k_{\text{max}} \) as the maximum trading frequency, let say \( k_{\text{max}} = 1 \) trade per day we can also define via a minimum trading time \( \tau_{\text{min}} = 1/k_{\text{max}} \) Considering such constraint, \( \sigma^2/\ud < k_{\text{max}} \) imposing a constraint on \( \ud \).

\[
\ud > \frac{\sigma^2}{k_{\text{max}}} \quad (41)
\]

However, this is just a rough approximation. The required trading rate is a random variable and if \( k_{\text{max}} \) is a hard condition, then we have to choose \( u \) and \( d \) so that \( ud >> \sigma^2/k_{\text{max}} \) with some confidence limit (confidence intervals can be estimated using the probability function of the trading time). That is setting \( ud \) so that the probability of not executing a trade in due time will not exceed a given small level: i.e. choose \( u \) and \( d \) such that \( \text{Probability}(\tau < \tau_{\text{min}}; u, d) = 1\% \) This will lead to large values \( u, d \) where the optimal conditions and approximations are no longer valid it is not possible to define such solution in close form, and complex optimization program are required, Such programs are implemented in YATS. In addition YATS provide setting to add other classical constraints for transaction costs.

The final objective is to get the highest wealth for a given time horizon ... If the initial wealth is \( W_0 \), then the terminal wealth \( W_0g(f)^n \) Considering this formula we have to maximize \( g(f)^n \) there are two ways :

- first increase the geometric mean per trade
- and/or increase the number of trades

Those 2 actions are not always compatible : increasing the expected geometric average will result in reducing the number of trades, and vice versa.

This is quite clear, if we put an order at, let say, 10% from the entry price, it is less likely to close the position during the day, since the probability of hitting the barriers are much smaller than the probability of hitting -1% or 2% barriers.

The general problem can be stated as follow: find \( f, u, d \) solution of maximize \( n\log(g(f)) \)

Negative returns.

The same program can be develop for short positions when the drift is negative. In this case, we have to consider the reverse of a positive drift strategy : sell at open price, set limit price at \(-d\%\), set stop price at \( u\%\)

In this case, the probability of a winning trade is the probability of hitting the \(-d\) barrier before the \( u \) barrier. this probability can be derived from the formula for a positive drift, mirroring the processes :

\[
P(\text{upper barrier bbefore lower barrier a}; m, a = \text{upper}; b = \text{lower})
= P(\text{lower barrier a before upper barrier b}; -m, b = \text{upper}; a = \text{lower}) \quad (42)
\]
1 References
